

CHAPTER - 3
ELECTROSTATICS

Chapter - 3

= ELECTROSTATICS =

Exercise 3.1 $P(2, \pi/2, -3) \Rightarrow P(0, 2, -3), Q(5, \pi, 0) \Rightarrow Q(-5, 0, 0)$

$$\vec{R} = \vec{PQ} = -5\vec{a}_x - 2\vec{a}_y + 3\vec{a}_z \quad R = \sqrt{38} \quad \frac{1}{4\pi\epsilon_0} = 9 \times 10^9$$

$$\vec{F}_{10nC} = - \frac{5 \times 10^{-9} \times 10 \times 10^{-9}}{38^{3/2}} \cdot 9 \times 10^9 [-5\vec{a}_x - 2\vec{a}_y + 3\vec{a}_z] = +9.61\vec{a}_x + 3.84\vec{a}_y - 5.76\vec{a}_z \text{ nN}$$

$$|\vec{F}_{10nC}| = 11.84 \text{ nN}$$

Exercise 3.2 $P(2, \pi/2, \pi/4) \Rightarrow P(\sqrt{2}, \sqrt{2}, 0), Q(1, \pi, \pi/2) \Rightarrow Q(0, 0, -1)$

$$\vec{R}_1 = \vec{QP} = 1.414\vec{a}_x + 1.414\vec{a}_y + \vec{a}_z \quad R_1 = \sqrt{3}$$

$$S(5, \pi/3, 2\pi/3) \Rightarrow S(-2.17, 3.75, 2.5)$$

$$\vec{R}_2 = \vec{SP} = 3.584\vec{a}_x - 2.336\vec{a}_y - 2.5\vec{a}_z \quad R_2 = 4.955$$

$$\vec{F}_{2,-0.5} = \frac{2 \times 10^{-9} \times -5 \times 10^{-9} \times 9 \times 10^9}{5^{3/2}} [\vec{R}_1] = -11.38\vec{a}_x - 11.38\vec{a}_y - 8.05\vec{a}_z \text{ nN}$$

$$\vec{F}_{2,0.2} = \frac{2 \times 10^{-9} \times 0.2 \times 10^{-9} \times 9 \times 10^9}{4.955^3} \vec{R}_2 = 0.106\vec{a}_x - 0.07\vec{a}_y - 0.074\vec{a}_z \text{ nN}$$

$$\vec{F}_2 = \vec{F}_{2,-0.5} + \vec{F}_{2,0.2} = -[11.274\vec{a}_x + 11.45\vec{a}_y + 8.124\vec{a}_z]$$

$$F_2 = 18 \text{ nN} \quad \text{force of attraction}$$

Exercise 3.3 $dq = 2\pi P' dP' z \Rightarrow dE = \frac{2\pi P' dP' z P_z}{4\pi\epsilon_0 [z^2 + P'^2]^{3/2}} \vec{a}_z$

$$\text{Hence } E_z = \frac{z P_z}{2\epsilon_0} \int_a^b \frac{P' dP'}{(P'^2 + z^2)^{3/2}} = \frac{P_z z}{2\epsilon_0} \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right]$$

Exercise 3.4 $|\vec{E}_+| = |\vec{E}_-| = \frac{P_z}{2\epsilon_0}$

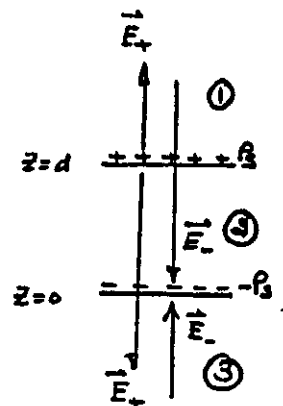
Region - 1: $\vec{E} = 0$

Region - 3: $\vec{E} = 0$

Region - 2:

$$\vec{E}_2 = - \left[\frac{P_z}{2\epsilon_0} + \frac{P_z}{2\epsilon_0} \right] \vec{a}_z$$

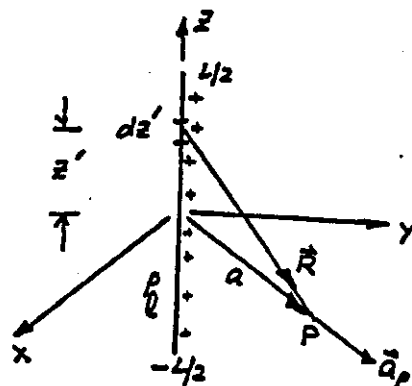
$$= - \frac{P_z}{\epsilon_0} \vec{a}_z \quad \text{V/m}$$



Exercise 3.5 $L = 10 \text{ m}$ $a = 5 \text{ m}$ $\rho_l = 10 \mu\text{C/m}$

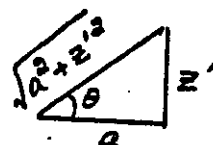
$$\vec{E} = \frac{\rho_l}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{(a\vec{a}_\rho - z'\vec{a}_z) dz'}{(a^2 + z'^2)^{3/2}}$$

$$E_\rho = \frac{2\rho_l}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{dz'}{(a^2 + z'^2)^{3/2}}$$



Let $z' = a \tan \theta$, $dz' = a \sec^2 \theta d\theta$, then

$$\int \frac{dz'}{(a^2 + z'^2)^{3/2}} = \frac{1}{a} \int \cos \theta d\theta = \frac{1}{a} \sin \theta = \frac{1}{a} \frac{z'}{\sqrt{a^2 + z'^2}}$$



$$E_\rho = \frac{\rho_l}{4\pi\epsilon_0 a} \left[\frac{L}{\sqrt{a^2 + (L/2)^2}} \right]$$

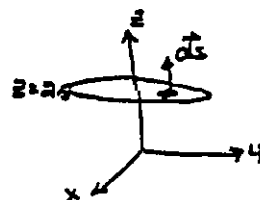
$$E_z = -\frac{\rho_l}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{z' dz'}{(a^2 + z'^2)^{3/2}} = \frac{\rho_l}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{a^2 + z'^2}} \right]_{-L/2}^{L/2} = 0$$

Hence, $\vec{E} = E_\rho \vec{a}_\rho$. Substitute values and obtain $\vec{E} = 25.46 \vec{a}_\rho \text{ kV/m}$

Exercise 3.6 $\vec{D} = 10 \sin \phi \vec{a}_\rho + 12z \cos(\phi/4) \vec{a}_z \text{ C/m}^2$

$d\vec{s} = \rho d\rho d\phi \vec{a}_z$ at $z = 2.5$

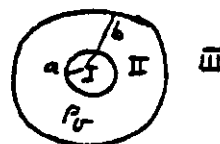
$$\Psi = \int \vec{D} \cdot d\vec{s} = 12 \times 2.5 \int_0^{0.5} \rho d\rho \int_0^{2\pi} \cos(\phi/4) d\phi = 15 \text{ C}$$



Exercise 3.7 $\vec{E}_I = 0$

$$\oint \vec{E} \cdot d\vec{s} = 4\pi r^2 E_r \quad a \leq r \leq b$$

$$Q = \int_a^r \frac{k}{r'} r'^2 dr' \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi = 2\pi k (r^2 - a^2)$$



Region - II $\vec{E}_{II} = \frac{k}{2\epsilon_0} \left(\frac{r^2 - a^2}{r^2} \right) \vec{a}_r$

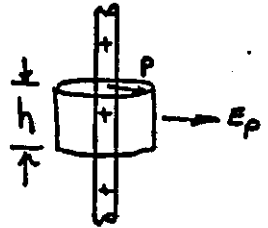
Region - III $Q = 2\pi k (b^2 - a^2) \Rightarrow \vec{E}_{III} = \frac{k}{2\epsilon_0} \left[\frac{b^2 - a^2}{r^2} \right] \vec{a}_r$

Exercise 3.8 $a \leq \rho \leq b$

$$Q = \int P_s ds = 2\pi a h P_s$$

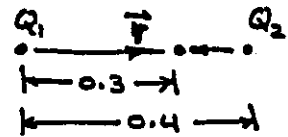
$$\oint \vec{D} \cdot d\vec{s} = 2\pi P h D_p \Rightarrow D_p = \frac{a P_s}{\rho} \Rightarrow E_p = \frac{a P_s}{\rho \epsilon_0}$$

$$V = \int \vec{D} \cdot d\vec{s} = a P_s \int_0^{2\pi} \frac{1}{\rho} d\phi \int_0^l dz = 2\pi a l P_s$$



Exercise 3.9 $Q_1 = 120 \text{ nC}, Q_2 = 800 \text{ nC}$

$$\vec{E} = \frac{Q_1}{4\pi\epsilon_0 r^2} \vec{a}_r$$



$$W = -Q_2 \int_C \vec{E} \cdot d\vec{l} = -\frac{Q_1 Q_2}{4\pi\epsilon_0} \int_{0.4}^{0.3} \frac{1}{r^2} dr = \frac{Q_1 Q_2}{4\pi\epsilon_0} \left[\frac{1}{0.3} - \frac{1}{0.4} \right] = 720 \mu\text{J}$$

Since $W > 0$, the external force is doing the work.

Exercise 3.10 $\vec{E} = \frac{Q}{4\pi\epsilon_0} \cdot \frac{\vec{a}_r}{r^2}$ but $\nabla(1/r) = -\frac{\vec{a}_r}{r^2}$

a) Thus, $\vec{E} = -\frac{Q}{4\pi\epsilon_0} \nabla(1/r) = -\nabla\left(\frac{Q}{4\pi\epsilon_0 r}\right) = -\nabla V$

b)

$$\nabla \times \vec{E} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta \vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ E_r & 0 & 0 \end{vmatrix} = 0$$

Exercise 3.11 $E_p = \frac{P_l}{2\pi\epsilon_0 \rho}$

$$V = -\int_{P_1}^{P_2} \vec{E} \cdot d\vec{l} = -\frac{P_l}{2\pi\epsilon_0} \int_{P_1}^{P_2} \frac{1}{\rho} d\rho = \frac{P_l}{2\pi\epsilon_0} \ln(P_1/P_2)$$

If we establish a reference point at $P_2 = \text{constant}$, then at $P_1 = \text{constant}$, $V = \text{constant}$. Thus, equipotential surfaces are concentric cylinders.

Exercise 3.12

$$V = \frac{P \cos \theta}{4\pi\epsilon_0 r^2} \quad \text{where } p = qd$$

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r} \vec{a}_r + \frac{1}{r} \frac{\partial V}{\partial \theta} \vec{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} \right] = \frac{P}{4\pi\epsilon_0 r^3} [2 \cos \theta \vec{a}_r + \sin \theta \vec{a}_\theta]$$

Exercise 3.13 Let $d\vec{l} = c\vec{E}$ where c is a constant

$$\begin{aligned} \text{Since } d\vec{l} &= dr \vec{a}_r + r d\theta \vec{a}_\theta + r \sin \theta d\phi \vec{a}_\phi \\ &= c E_r \vec{a}_r + c E_\theta \vec{a}_\theta, \end{aligned}$$

$$dr = \frac{2P \cos \theta}{4\pi\epsilon_0 r^3} c, \quad r d\theta = \frac{P \sin \theta}{4\pi\epsilon_0 r^3} c \Rightarrow \frac{dr}{r} = \frac{2 \cos \theta d\theta}{\sin \theta} \quad \text{or}$$

$$2 \ln(\sin \theta) = \ln(r) + \ln(k) \quad \text{where } k \text{ is a constant of integration}$$

$$\text{or } \sin^2 \theta = rk \Rightarrow r = d \sin^2 \theta \quad \text{where } d = \frac{1}{k}$$

Thus, $r = d \sin^2 \theta$ yields the lines of force of a dipole. These are the curves shown in Figure 3.26.

Exercise 3.14 $E_r = \frac{P}{4\pi\epsilon_0 r^3} 2 \cos \theta, \quad E_\theta = \frac{P}{4\pi\epsilon_0 r^3} \sin \theta$

$$E = \sqrt{E_r^2 + E_\theta^2} = \frac{P}{4\pi\epsilon_0 r^3} \sqrt{4 \cos^2 \theta + \sin^2 \theta} = \frac{P}{4\pi\epsilon_0 r^3} \sqrt{1 + 3 \cos^2 \theta}$$

Exercise 3.15

$$\nabla \times \vec{E} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \vec{a}_r & r \vec{a}_\theta & r \sin \theta \vec{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{2P \cos \theta}{4\pi\epsilon_0 r^3} & \frac{P \sin \theta}{4\pi\epsilon_0 r^3} & 0 \end{vmatrix}$$

$$= \frac{1}{r} \vec{a}_\phi \left[-\frac{2P \sin \theta}{4\pi\epsilon_0 r^3} + \frac{2P \sin \theta}{4\pi\epsilon_0 r^3} \right] = 0$$

Since $\nabla \times \vec{E} = 0$, \vec{E} represents a conservative field.

Exercise 3.16

Region-1: $\vec{E}_1 = 0$

Region-2: $\vec{E}_2 = \frac{Q}{2\pi\epsilon_0 r} \vec{a}_r$ as $r \leq b$

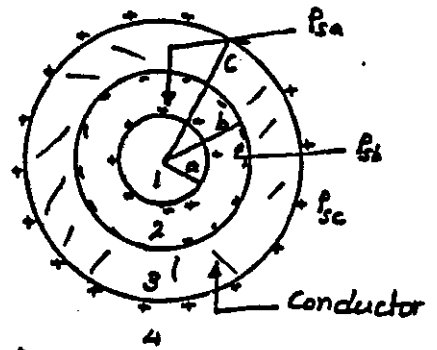
Charge per unit length: $Q = 2\pi a \rho_{sa}$

at $r = b$, $\rho_{sb} = -\frac{Q}{2\pi b} = -\frac{a}{b} \rho_{sa}$

at $r = c$, $\rho_{sc} = \frac{a}{c} \rho_{sa}$

Region-3: $\vec{E}_3 = 0$ Conductor, Region-4: $\vec{E}_4 = \frac{Q}{2\pi\epsilon_0 r} \vec{a}_r$, $r > c$

When the outer conductor is grounded, $\rho_{sc} \rightarrow 0$ and $\vec{E}_4 \rightarrow 0$.



Exercise 3.17

$$V_{ab} = - \int_b^a \vec{E}_2 \cdot d\vec{l} = - \frac{Q}{2\pi\epsilon_0} \int_b^a \frac{1}{r} dr = \frac{a \rho_{sa}}{\epsilon_0} \ln(b/a)$$

Potential at a is higher than at b .

Exercise 3.18

$\rho_v = b^2 - r^2$ C/m³ $\oint_S \vec{E} \cdot d\vec{S} = 4\pi r^2 E_r$

Region-1: $r \leq b$ $Q_{enc} = \int_0^r (b^2 - r^2) r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$

$$= \frac{4\pi r^3}{15} (5b^2 - 3r^2)$$

Thus, $\vec{E}_1 = \frac{(5b^2 - 3r^2)r}{15\epsilon_0} \vec{a}_r$ $r \leq b$

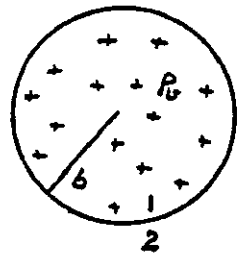
Region-2: $r \geq b$: $Q_{enc} = \frac{8\pi b^5}{15} \Rightarrow \vec{E}_2 = \frac{2b^5}{15\epsilon_0 r^2} \vec{a}_r$

Potential at any point at $r \geq b$:

$$V_r = - \int_{\infty}^r \frac{2b^5}{15\epsilon_0 r^2} dr = \frac{2b^5}{15\epsilon_0} \cdot \frac{1}{r}$$

at $r = \infty$ $V_{\infty} \rightarrow 0$, Reference point

at $r = b$, $V_b = \frac{2b^4}{15\epsilon_0}$



Potential inside the sphere:

$$V_r' = - \int \vec{E}_1 \cdot d\vec{l} = - \frac{5b^2}{15\epsilon_0} \left(\frac{r^2}{2} \right) + \frac{3r^4}{60\epsilon_0} + k$$

where k is an integration constant

at $r = b$, $V_r' = V_b \Rightarrow k = \frac{b^4}{4\epsilon_0}$

Hence

$$V_r' = \frac{b^4}{4\epsilon_0} - \frac{5b^2 r^2}{30\epsilon_0} + \frac{3r^4}{60\epsilon_0}$$

Exercise 3.19

$$\vec{R} = (x-x')\vec{a}_x + (y-y')\vec{a}_y + (z-z')\vec{a}_z$$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\frac{\partial}{\partial x} \left(\frac{1}{R} \right) = \frac{x-x'}{R^3}$$

$$\nabla' \left(\frac{1}{R} \right) = \frac{\partial}{\partial x'} \left(\frac{1}{R} \right) \vec{a}_x + \frac{\partial}{\partial y'} \left(\frac{1}{R} \right) \vec{a}_y + \frac{\partial}{\partial z'} \left(\frac{1}{R} \right) \vec{a}_z$$

$$= \frac{(x-x')}{R^3} \vec{a}_x + \frac{(y-y')}{R^3} \vec{a}_y + \frac{(z-z')}{R^3} \vec{a}_z = \frac{\vec{R}}{R^3} = \frac{\vec{a}_R}{R^2}$$

Exercise 3.20

$$A = \pi a^2 = \pi (10 \times 10^{-3})^2 = 314.16 \times 10^{-6} \text{ m}^2$$

$$\vec{P} = 2z^2 + 10 \vec{a}_z$$

$$P_{vb} = -\nabla \cdot \vec{P} = -4z$$

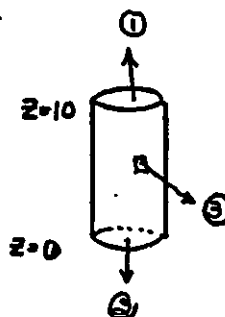
$$\text{Surface-1: } P_{sb1} = \vec{P} \cdot \vec{a}_z \Big|_{z=10} = 2z^2 + 10 = 210 \text{ C/m}^2$$

$$\text{Surface-2: } P_{sb2} = -\vec{P} \cdot \vec{a}_z \Big|_{z=0} = -10 \text{ C/m}^2$$

$$\text{Surface-3: } P_{sb3} = \vec{P} \cdot \vec{a}_r = 0$$

$$\text{bound volume charge: } \int_V P_{vb} dv = \pi a^2 \int_0^{10} -4z dz = -200 \pi a^2$$

$$\text{Total bound charge: } 210 \pi a^2 - 10 \pi a^2 - 200 \pi a^2 = 0 \quad (\text{As expected})$$



bound charge on surface-1: $210 \pi a^2$

bound charge on surface-2: $-10 \pi a^2$

Exercise 3.21

$$\vec{P} = x \vec{a}_x + y \vec{a}_y + z \vec{a}_z$$

$$\nabla \cdot \vec{P} = 3 \Rightarrow P_{sv} = -\nabla \cdot \vec{P} = -3 \text{ C/m}^3$$

$$\text{Surface at } \quad \text{bound } P_{sb} \quad \text{Total Charge (bound)}$$

$$x = b/2 \quad b/2 \quad b^3/2$$

$$x = -b/2 \quad b/2 \quad b^3/2$$

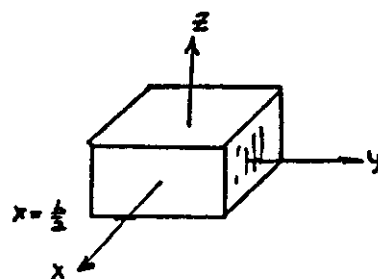
$$y = b/2 \quad b/2 \quad b^3/2$$

$$y = -b/2 \quad b/2 \quad b^3/2$$

$$z = b/2 \quad b/2 \quad b^3/2$$

$$z = -b/2 \quad b/2 \quad b^3/2$$

$$\text{Total bound surface charge} = \frac{3b^3}{2}$$



Total bound volume charge: $-3b^3$

Hence, net bound charge = 0

Exercise 3.22 $\vec{P} = P \vec{a}_z$

$$\nabla \cdot \vec{P} = 0 \Rightarrow P_{vb} = 0$$

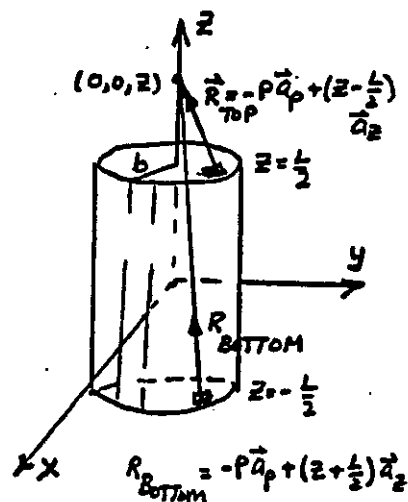
$$P_{sb}|_{z=\frac{L}{2}} = \vec{P} \cdot \vec{a}_z = P, \quad P_{sb}|_{z=-\frac{L}{2}} = -P$$

$$\vec{E}_{TOP} = \frac{P}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^b \frac{P d\rho d\phi (z - \frac{L}{2}) \vec{a}_z}{[\rho^2 + (z - \frac{L}{2})^2]^{3/2}}$$

$$= -\frac{P}{2\epsilon_0} \left[\frac{z - \frac{L}{2}}{\sqrt{b^2 + (z - \frac{L}{2})^2}} - 1 \right] \vec{a}_z$$

$$\vec{E}_{BOTTOM} = \frac{P}{2\epsilon_0} \left[\frac{z + \frac{L}{2}}{\sqrt{b^2 + (z + \frac{L}{2})^2}} - 1 \right] \vec{a}_z$$

$$\vec{E} = \vec{E}_{TOP} + \vec{E}_{BOTTOM} = \frac{P}{2\epsilon_0} \left[\frac{z + \frac{L}{2}}{\sqrt{b^2 + (z + \frac{L}{2})^2}} + \frac{z - \frac{L}{2}}{\sqrt{b^2 + (z - \frac{L}{2})^2}} \right] \vec{a}_z$$



Exercise 3.23 $\oint_S \vec{D} \cdot d\vec{s} = Q_{enc} \Rightarrow 2\pi r L D_r = P_L L$ or $D_r = \frac{P_L}{2\pi r}$

$$\text{and } E_r = \frac{D_r}{\epsilon} = \frac{P_L}{2\pi \epsilon r} \quad \epsilon = \epsilon_0 \epsilon_r$$

$$\text{Since } D_r = \epsilon_0 E_r + P_r \Rightarrow P_r = \frac{P_L}{2\pi r} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right) \text{ or } \vec{P} = \frac{P_L}{2\pi r} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right) \vec{a}_r$$

$$\nabla \cdot \vec{P} = 0 \Rightarrow P_{vb} = 0$$

$$\text{For a } \rho \text{ be the radius of the line, } P_{sb} = -\vec{P} \cdot \vec{a}_\rho = -\left(\frac{\epsilon_r - 1}{\epsilon_r} \right) \frac{P_L}{2\pi a}$$

$$Q_{sb} | \text{ unit length (bound) } = 2\pi a P_{sb} = -\left(\frac{\epsilon_r - 1}{\epsilon_r} \right) P_L$$

Hence, the effective line charge density

$$P_e = P_L - \frac{\epsilon_r - 1}{\epsilon_r} P_L = \frac{P_L}{\epsilon_r}$$

The line charge has decreased by a factor of ϵ_r . Note that ϵ_r has no effect on the \vec{D} field.

Exercise 3.24 From Gauss' Law: $D_r = \frac{\rho_v}{3} r \quad r \leq a$
 $= \frac{\rho_v}{3} \frac{a^3}{r^2} \quad r \geq a$

for $r > a$ $V_o = - \int_{-\infty}^r \vec{E} \cdot d\vec{l} = - \frac{\rho_v}{3\epsilon_0} a^3 \int_{-\infty}^r \frac{1}{r^2} dr = \frac{\rho_v}{3\epsilon_0} \frac{a^3}{r} \Rightarrow V_a = \frac{\rho_v}{3\epsilon_0} a^2 \text{ at } r=a$

for $r \leq a$ $V_r - V_a = - \int_a^r \frac{\rho_v}{3\epsilon_0} r dr = - \frac{\rho_v}{3\epsilon_0} \left[\frac{r^2}{2} \right]_a^r = - \frac{\rho_v}{6\epsilon_0} [r^2 - a^2]$

Hence, inside the sphere, $V_r = \frac{\rho_v}{3\epsilon_0} a^2 - \frac{\rho_v}{6\epsilon_0} (r^2 - a^2) = \frac{\rho_v a^2}{2\epsilon_0} - \frac{\rho_v r^2}{6\epsilon_0}$

$W = \frac{1}{2} \int_V \rho_v V_r dv = \frac{1}{2} \rho_v^2 \int_0^a \left(\frac{a^2}{2\epsilon_0} - \frac{r^2}{6\epsilon_0} \right) r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{4\pi a^5 \rho_v^2}{15\epsilon_0}$

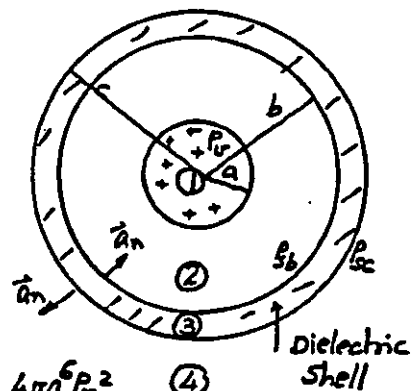
Exercise 3.25 $W = \frac{1}{2} \int_V \vec{D} \cdot \vec{E} dv = \frac{\rho_v^2}{2} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^a \epsilon_0 \left(\frac{r}{3\epsilon_0} \right)^2 r^2 dr$
 $+ \frac{\rho_v^2}{2} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_a^\infty \epsilon_0 \left(\frac{a^3}{3\epsilon_0 r^2} \right)^2 r^2 dr = \frac{4\pi a^5 \rho_v^2}{15\epsilon_0}$

Exercise 3.26

$D_r = \frac{\rho_v}{3} r \quad r \leq a$
 $= \frac{\rho_v}{3} \frac{a^3}{r^2} \quad r \geq a$

Region-1; $r \leq a$

$W_1 = \frac{1}{2\epsilon_0} \int_0^a \left(\frac{\rho_v}{3} r \right)^2 r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{4\pi a^5 \rho_v^2}{90\epsilon_0}$



Region-2: $a \leq r \leq b$: $W_2 = \frac{4\pi}{2\epsilon_0} \int_a^b \left(\frac{\rho_v a^3}{3r^2} \right)^2 r^2 dr = \frac{4\pi a^5 \rho_v^2}{18\epsilon_0} - \frac{4\pi a^6 \rho_v^2}{18b\epsilon_0}$

Region-3: $b \leq r \leq c$: $W_3 = \frac{4\pi}{2\epsilon} \int_b^c \left(\frac{\rho_v a^3}{3r^2} \right)^2 r^2 dr = \frac{4\pi a^6 \rho_v^2}{18b\epsilon} - \frac{4\pi a^6 \rho_v^2}{18\epsilon c}$

Region-4, $c \leq r \leq \infty$ $W_4 = \frac{4\pi}{2\epsilon_0} \int_c^\infty \left(\frac{\rho_v a^3}{3r^2} \right)^2 r^2 dr = \frac{4\pi a^6 \rho_v^2}{18\epsilon_0 c}$

$W = W_1 + W_2 + W_3 + W_4 = \frac{4\pi a^5 \rho_v^2}{18\epsilon_0} \left[\frac{\epsilon}{3} + \frac{\epsilon_r - 1}{\epsilon_r} \left(\frac{a}{c} - \frac{a}{b} \right) \right]$

$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \frac{\rho_v}{3} \frac{a^3}{r^2} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right) \vec{a}_r \quad \nabla \cdot \vec{P} = 0 \Rightarrow \rho_{vb} = 0$

At $r=b$ surface: $P_{sb}|_{r=b} = - \vec{P} \cdot \vec{a}_r = - \frac{\rho_v}{3} \frac{a^3}{b^2} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right)$

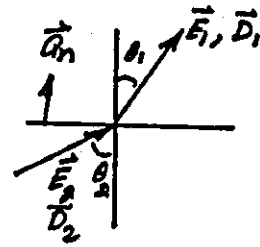
At $r=c$ surface: $P_{sc}|_{r=c} = \frac{\rho_v}{3} \frac{a^3}{c^2} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right)$

Exercise 3.27 Dielectric Media $\Rightarrow P_s = 0$

$$E_{t1} = E_{t2} \Rightarrow E_1 \sin \theta_1 = E_2 \sin \theta_2 \quad (1)$$

$$D_{n1} = D_{n2} \Rightarrow \epsilon_1 E_1 \cos \theta_1 = \epsilon_2 E_2 \cos \theta_2 \quad (2)$$

From (1) and (2): $\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}}$



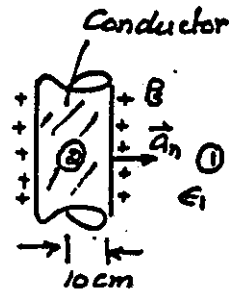
Exercise 3.28 $P_s = 200 \mu\text{C}/\text{m}^2$ $\vec{a}_n = \vec{a}_p$

$\vec{D}_2 = 0$ and $\vec{E}_2 = 0$ Conducting region. Thus.

$$D_{n1} = D_{p1} = P_s = 200 \mu\text{C}/\text{m}^2$$

$$E_{p1} = \frac{D_{p1}}{\epsilon_1} = \frac{200 \times 10^{-6}}{5 \times 10^{-9}} \cdot 36\pi = 4.52 \text{ MV/m}$$

$$P_{s1} = D_{p1} - \epsilon_0 E_{p1} = 160 \mu\text{C}/\text{m}^2 \Rightarrow P_{sb} \Big|_{P=10\text{cm}} = -160 \mu\text{C}/\text{m}^2$$

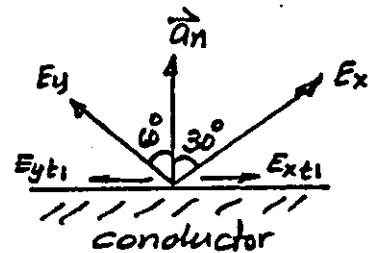


Exercise 3.29 $E_x = 10 \text{ V/m}$

$$E_{xt1} = 10 \sin 30^\circ = 5 \text{ V/m}$$

$$E_{yt1} = E_y \sin 60^\circ = 0.866 E_y \text{ V/m}$$

At the boundary, $E_{xt1} = E_{yt1} \Rightarrow E_y = 5.77 \text{ V/m}$



Exercise 3.30 $C = \frac{\epsilon A}{d} = \frac{6 \times 10^{-9}}{36\pi} \cdot \frac{40 \times 10^{-4}}{2 \times 10^{-3}} = 106.1 \text{ pF}$

$$\vec{E} = -\frac{V}{d} \vec{a}_z = -\frac{1500}{2 \times 10^{-3}} \vec{a}_z = -750 \vec{a}_z \text{ kV/m}, \quad \vec{D} = \epsilon \vec{E} = -\frac{6 \times 10^{-6}}{36\pi} \cdot 750 \times 10^3 \vec{a}_z$$

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} = \vec{D} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right) = -33.16 \vec{a}_z \mu\text{C}/\text{m}^2 = -39.79 \vec{a}_z \mu\text{C}/\text{m}^2$$

$$P_{s+} = 39.79 \mu\text{C}/\text{m}^2 \text{ (Top-plate)}, \quad P_{s-} = -39.79 \mu\text{C}/\text{m}^2 \text{ (bottom plate)}$$

$$\text{Bound charges: } P_{sb \text{ Top}} = -33.16 \mu\text{C}/\text{m}^2, \quad P_{sb \text{ Bottom}} = 33.16 \mu\text{C}/\text{m}^2$$

$$\text{Energy density: } w = \frac{1}{2} \epsilon E^2 = \frac{1}{2} \times \frac{6 \times 10^{-9}}{36\pi} \cdot (750 \times 10^3)^2 = 14.92 \text{ J/m}^3$$

$$\text{Total Energy: } W = \int w dv = 14.92 \times 40 \times 10^{-4} \times 2 \times 10^{-3} = 119.37 \mu\text{J}$$

Exercise 3.31 $D_r = \frac{Q}{r} \epsilon \Rightarrow E_r = \frac{Q}{r \epsilon} \cdot V_{ab} = -\int_a^b \frac{Q}{\epsilon r} dr = \frac{Q}{\epsilon} \ln(b/a)$

$$C = \frac{Q}{V_{ab}} = \frac{(2\pi a L \epsilon_r) \epsilon}{a \epsilon_r \ln(b/a)} = \frac{2\pi \epsilon L}{\ln(b/a)}$$

Exercise 3.32 $\frac{1}{C} = \int_a^b \frac{dr}{2\pi r L \epsilon} = \frac{1}{2\pi \epsilon L} \ln(b/a) \Rightarrow C = \frac{2\pi \epsilon L}{\ln(b/a)}$

Exercise 3.33 $V_{ab} = 1000 \text{ V}$ $a = 0.1 \text{ m}$ $b = 0.12 \text{ m}$ $\epsilon_r = 2.5$

$$V_{ab} = \frac{Q}{4\pi \epsilon} \left[\frac{1}{a} - \frac{1}{b} \right] \Rightarrow Q = \frac{1000 \times 4\pi \times 8.5 \times 10^{-9}}{36\pi} \frac{(0.1)(0.12)}{0.12 - 0.1} = 166.67 \text{ nC}$$

$$E_r = \frac{Q}{4\pi \epsilon r^2} = \frac{166.67 \times 10^{-9}}{r^2} \cdot \frac{9 \times 10^9}{2.5} = \frac{600}{r^2} \text{ V/m}$$

$$D_r = \epsilon E_r = \frac{600}{r^2} \cdot \frac{8.5 \times 10^{-9}}{36\pi} = \frac{13.26}{r^2} \text{ nC/m}^2 \Rightarrow P_r = D_r - \epsilon_0 E_r = \frac{7.96}{r^2} \text{ nC/m}^2$$

Free charge densities: $P_{sa} = \frac{Q}{4\pi a^2} = 1.326 \mu\text{C/m}^2$, $P_{sb} = -\frac{Q}{4\pi b^2} = -92.1 \text{ nC/m}^2$

bound charge densities: $P_{ub} = -\nabla \cdot \vec{P} = 0$ $P_{sb} = \vec{P} \cdot \vec{a}_n \Rightarrow$

at $r=a$, $P_{sb}|_{r=a} = -\frac{7.96 \times 10^{-9}}{100 \times 10^{-4}} = -796 \text{ nC/m}^2$, $P_{sb}|_{r=b} = +\frac{7.96 \times 10^{-9}}{144 \times 10^{-4}} = 552.78 \text{ nC/m}^2$

$$C = \frac{Q}{V_{ab}} = 166.67 \text{ pF}$$

Exercise 3.34 $A = 40 \times 10^{-4} \text{ m}^2$ $d = 2 \times 10^{-3} \text{ m}$

$$\vec{E} = -750 \vec{a}_z \text{ kV/m} \quad \vec{D} = \epsilon_0 \vec{E} = -6.63 \vec{a}_z \mu\text{C/m}^2 \quad C = \frac{\epsilon_0 A}{d} = 17.68 \text{ pF}, \quad \vec{P} = 0$$

$$P_{s\text{top}} = 6.63 \mu\text{C/m}^2, \quad P_{s\text{bottom}} = -6.63 \mu\text{C/m}^2, \quad w = \frac{1}{2} \vec{D} \cdot \vec{E} = 2.487 \text{ J/m}^3$$

$$W = \int w dv = 2.487 \times 40 \times 10^{-4} \times 2 \times 10^{-3} = 19.89 \mu\text{J}$$

Exercise 3.35 $Q = \frac{V_{ab} (4\pi \epsilon_0) ab}{b-a} = \frac{1000 \times 0.1 \times 0.12}{9 \times 10^9 (0.12 - 0.1)} = 66.67 \text{ nC}$

$$E_r = \frac{Q}{4\pi \epsilon_0 r^2} = \frac{600}{r^2} \text{ V/m}, \quad D_r = \epsilon_0 E_r = \frac{5.31}{r^2} \text{ nC/m}^2$$

$$P_s|_{r=a} = \frac{66.67 \times 10^{-9}}{4\pi \times 100 \times 10^{-4}} = 530.54 \text{ nC/m}^2$$

$$C = \frac{Q}{V_{ab}} = \frac{66.67 \times 10^{-9}}{1000}$$

$$P_s|_{r=b} = -\frac{66.67 \times 10^{-9}}{4\pi \times 144 \times 10^{-4}} = -368.43 \text{ nC/m}^2$$

$$= 66.67 \text{ pF} \quad \epsilon_{\text{dielectric}} / \epsilon_{\text{air}} = 2.5$$

Exercise 3.36

$$\epsilon = \epsilon_1 + \frac{\epsilon_2 - \epsilon_1}{d} z \quad \epsilon = \epsilon_1 \text{ at } z=0, \quad \epsilon = \epsilon_2 \text{ at } z=d$$

$$\frac{1}{C} = \int_0^d \frac{dz}{(\epsilon_1 + \frac{\epsilon_2 - \epsilon_1}{d} z) A} = \frac{d}{A(\epsilon_2 - \epsilon_1)} \ln(\epsilon_2 / \epsilon_1)$$

A = Area

$$\text{Hence } C = \frac{A(\epsilon_2 - \epsilon_1)}{d \ln(\epsilon_2 / \epsilon_1)} \quad \text{①}$$

$$\text{When } \epsilon_2 \rightarrow \epsilon_1, \quad e^{\ln(\epsilon_2 / \epsilon_1)} \approx 1 + \ln(\frac{\epsilon_2}{\epsilon_1}) \Rightarrow \ln(\frac{\epsilon_2}{\epsilon_1}) = \frac{\epsilon_2 - \epsilon_1}{\epsilon_1}$$

$$\text{Hence from ① } C = \frac{A \epsilon_1}{d}$$

Exercise 3.37

$$\nabla^2 V = 0 \Rightarrow \frac{\partial}{\partial r} (r^2 \frac{\partial V}{\partial r}) = 0 \Rightarrow V = -\frac{C_1}{r} + C_2$$

$$\text{at } r=b, V=0 \Rightarrow C_2 = \frac{C_1}{b}. \text{ Thus, } V = -C_1 [\frac{1}{r} - \frac{1}{b}]$$

$$\text{at } r=a, V=V_0 \Rightarrow C_1 = \frac{V_0}{\frac{1}{a} - \frac{1}{b}}. \text{ Finally, } V = \frac{V_0}{\frac{1}{a} - \frac{1}{b}} [\frac{1}{r} - \frac{1}{b}]$$

$$\vec{E} = -\nabla V = -\frac{C_1}{r^2} \vec{a}_r = \frac{ab V_0}{(b-a)r^2} \vec{a}_r \quad D_r = \frac{\epsilon V_0 ab}{(b-a)r^2} \quad \rho_{sa} = \frac{\epsilon V_0 ab}{(b-a)a^2}, \rho_{sb} = \frac{-\epsilon V_0 ab}{(b-a)b^2}$$

$$Q_a = \frac{\epsilon V_0 ab}{(b-a)a^2} \cdot 4\pi a^2 = \frac{4\pi \epsilon V_0 ab}{(b-a)}, \quad C = \frac{Q_a}{V_0} = \frac{4\pi \epsilon ab}{b-a}, \quad W = \frac{1}{2} C V_0^2 = \frac{2\pi \epsilon ab}{b-a} V_0^2$$

Exercise 3.38

L = Length

$$\nabla^2 V = 0 \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial V}{\partial r}) = 0$$

$$\text{Region-1: } V_1 = k_1 \ln r + k_2$$

$$\text{at } r=a, V_1 = V_0 \Rightarrow k_2 = V_0 - k_1 \ln a$$

$$V_1 = V_0 + k_1 \ln(r/a)$$

$$E_{1r} = -\frac{\partial V_1}{\partial r} = -\frac{k_1}{r}, \quad D_{1r} = -\frac{\epsilon_1 k_1}{r}$$

$$\text{at } r=c, V_1 = V_2 \text{ and } D_{1r} = D_{2r}$$

$$\text{Thus, } V_0 + k_1 \ln(\frac{c}{a}) = k_3 \ln(c/b)$$

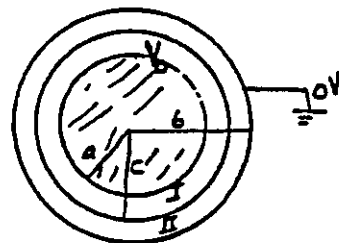
$$\text{and } \epsilon_1 k_1 = \epsilon_2 k_3$$

$$k_1 = \frac{-\epsilon_2 V_0}{M}, \quad k_3 = \frac{-\epsilon_1 V_0}{M}$$

$$\text{where } M = \epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)$$

$$P_s|_{r=a} = D_{1r}|_{r=a} = -\frac{\epsilon_1 k_1}{a} = \frac{\epsilon_1 \epsilon_2 V_0}{a M}$$

$$Q_a = 2\pi a L P_s|_{r=a} = \frac{2\pi \epsilon_1 \epsilon_2 V_0 L}{M}$$



$$\text{Region-2: } V_2 = k_3 \ln r + k_4$$

$$\text{at } r=b, V_2 = 0 \Rightarrow k_4 = -k_3 \ln b$$

$$V_2 = k_3 \ln(r/b)$$

$$E_{2r} = -\frac{\partial V_2}{\partial r} = -\frac{k_3}{r}, \quad D_{2r} = -\frac{\epsilon_2 k_3}{r}$$

$$C = \frac{Q_a}{V_0} = \frac{2\pi \epsilon_1 \epsilon_2 L}{\epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)}$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \text{ where}$$

$$C_1 = \frac{2\pi \epsilon_1 L}{\ln(c/a)}, \quad C_2 = \frac{2\pi \epsilon_2 L}{\ln(b/a)}$$

C_1 and C_2 are connected in series.

Exercise 3.39 $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

$\frac{\partial V}{\partial x}: \frac{\partial^2}{\partial x^2} \left(\frac{\partial V}{\partial x} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial V}{\partial x} \right) + \frac{\partial^2}{\partial z^2} \left(\frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial x} (\nabla^2 V) = 0$

Similarly $\frac{\partial^2 V}{\partial x^2}$ is a solution because $\frac{\partial^2}{\partial x^2} (\nabla^2 V) = 0$

Finally $\frac{\partial^2}{\partial x \partial y} (\nabla^2 V) = 0 \Rightarrow \frac{\partial^2 V}{\partial x \partial y}$ is also a solution

Exercise 3.40 For $0 \leq z \leq d$, $\nabla \cdot \vec{D} = 0 \Rightarrow \nabla \cdot \epsilon \vec{E} = 0 \Rightarrow \epsilon \nabla \cdot \vec{E} + \vec{E} \cdot \nabla \epsilon = 0$

$\epsilon = \epsilon_0(1+mz) \Rightarrow \nabla \epsilon = m\epsilon_0 \vec{a}_z$. Thus, $\epsilon \frac{\partial E_z}{\partial z} + m\epsilon_0 E_z = 0$

Or $\frac{dE_z}{E_z} = -\frac{m}{1+mz} dz \Rightarrow \ln(E_z) = -\ln(1+mz) + \ln A$, $\ln A = \text{Integration constant}$

Thus, $E_z = \frac{A}{1+mz} \Rightarrow -\frac{dV}{dz} = \frac{A}{1+mz} \Rightarrow V = -\frac{A}{m} \ln(1+mz) + B$, $B = \text{constant of Integration}$

at $z=0$ $V=0 \Rightarrow B=0$ and at $z=d$, $V=V_0 \Rightarrow A = -\frac{V_0 m}{\ln(1+md)}$

Hence, $V = V_0 \frac{\ln(1+mz)}{\ln(1+md)}$ and $E_z = -\frac{V_0 m}{(1+mz) \ln(1+md)}$

$D_z = -\frac{\epsilon V_0 m}{(1+mz) \ln(1+md)} = -\frac{V_0 m \epsilon_0}{\ln(1+md)}$ $P_S|_{z=d} = \frac{V_0 m \epsilon_0}{\ln(1+md)}$

$Q|_{z=d} = \frac{V_0 m \epsilon_0 S}{\ln(1+md)}$ and $C = \frac{m \epsilon_0 S}{\ln(1+md)}$ where $S = \text{Area of each plate}$

Exercise 3.41

Let the potential at $P=b$ is zero. Then

$V_+ = \frac{P_1}{2\pi\epsilon} \ln(b/r_1)$ $V_- = -\frac{P_1}{2\pi\epsilon} \ln(b/r_2)$

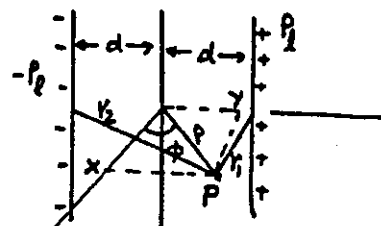
$V_P = V_+ + V_- = \frac{P_1}{2\pi\epsilon} \ln(r_2/r_1)$

Equipotential surfaces at

$\frac{r_2}{r_1} = k$ where k is a constant

or $(d+y)^2 + x^2 = k^2 [(d-y)^2 + x^2] \Rightarrow x^2 + y^2 - 2yd \frac{k^2+1}{k^2-1} + d^2 = 0$

Add $d^2 \left[\frac{k^2+1}{k^2-1} \right]^2$ to both sides to complete squares,



$r_1 = \sqrt{P^2 + d^2 - 2Pd \sin \phi} = \sqrt{(d-y)^2 + x^2}$

$r_2 = \sqrt{P^2 + d^2 + 2Pd \sin \phi} = \sqrt{(d+y)^2 + x^2}$

$$x^2 + \left(y - d \frac{k^2+1}{k^2-1}\right)^2 = \left(\frac{2kd}{k^2-1}\right)^2$$

This equation describes a series of circles (as shown) in the xy plane.

To determine surface charge density, let us compute normal component (\vec{a}_y) of \vec{D} -field.

$$E_y = -\frac{\partial V}{\partial y} = -\frac{\rho_0}{2\pi\epsilon} \left[\frac{1}{r_2} \frac{\partial r_2}{\partial y} - \frac{1}{r_1} \frac{\partial r_1}{\partial y} \right]$$

$$= -\frac{\rho_0}{2\pi\epsilon} \left[\frac{d+y}{r_2^2} + \frac{d-y}{r_1^2} \right] \Rightarrow D_y = -\frac{\rho_0}{2\pi} \left[\frac{d+y}{r_2^2} + \frac{d-y}{r_1^2} \right]$$

$$\rho_s|_{y=0} = -\frac{\rho_0 d}{\pi(d^2+x^2)}$$

Thus, the charge upon the infinite conductor per meter of length along the line is

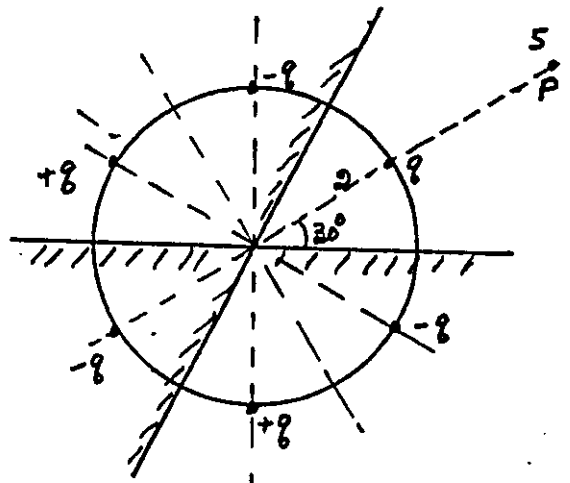
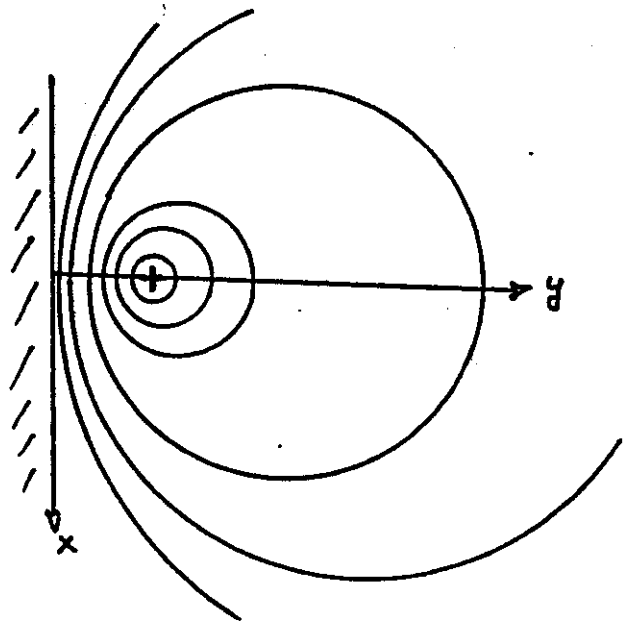
$$Q = -\frac{\rho_0 d}{\pi} \int_{-\infty}^{\infty} \frac{dx}{d^2+x^2} \int_0^1 dy = -\frac{\rho_0}{\pi} \left[\tan^{-1} \frac{x}{d} \right]_{-\infty}^{\infty} = -\rho_0 \text{ C/m.}$$

Exercise 3.42

The two intersecting planes can be replaced by image charges as shown.

$$V_P = \frac{q}{4\pi\epsilon} \left[\frac{1}{3} - \frac{1}{7} - \frac{1}{4.36} + \frac{1}{6.25} + \frac{1}{6.25} - \frac{1}{4.36} \right]$$

$$= \frac{0.52q}{4\pi\epsilon}$$



Problem 3.1 $\vec{R} = \vec{QP} = -3\vec{a}_x + 4\vec{a}_y$, $R = 5$

$$\vec{F}_P = \vec{F}_{\mu C} = \frac{2 \times 10^{-6} \times 10 \times 10^{-6}}{5^3} \cdot 10^9 \times 9 [-3\vec{a}_x + 4\vec{a}_y] = -4.32\vec{a}_x + 5.76\vec{a}_y \text{ mN}$$

$$\vec{F}_Q = 4.32\vec{a}_x - 5.76\vec{a}_y \text{ mN}, \quad |\vec{F}_P| = |\vec{F}_Q| = 7.2 \text{ mN}$$

Problem 3.2 $\vec{R}_1 = -0.2\vec{a}_x - 0.3\vec{a}_y \Rightarrow R_1 = 0.36$

$$\vec{R}_2 = -0.5\vec{a}_x - 0.7\vec{a}_y + 1.3\vec{a}_z, \quad R_2 = 1.56$$

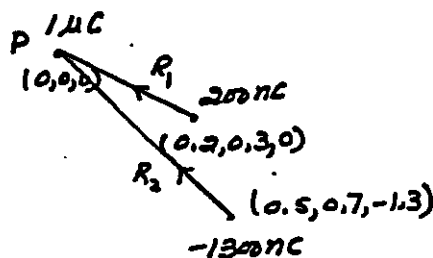
$$\vec{F}_{P1} = \frac{1 \times 10^{-6} \times 200 \times 10^{-9}}{0.36^3} \cdot 9 \times 10^9 [\vec{R}_1]$$

$$= -7.72\vec{a}_x - 11.57\vec{a}_y \text{ mN}$$

$$\vec{F}_{P2} = -\frac{1 \times 10^{-6} \times 1300 \times 10^{-9}}{1.56^3} \cdot 9 \times 10^9 \vec{R}_2$$

$$= 1.54\vec{a}_x + 2.16\vec{a}_y - 4.01\vec{a}_z \text{ mN}$$

$$\vec{F}_P = \vec{F}_{P1} + \vec{F}_{P2} = -6.18\vec{a}_x - 9.41\vec{a}_y - 4.01\vec{a}_z \text{ mN}, \quad |\vec{F}_P| = 11.95 \text{ mN}$$



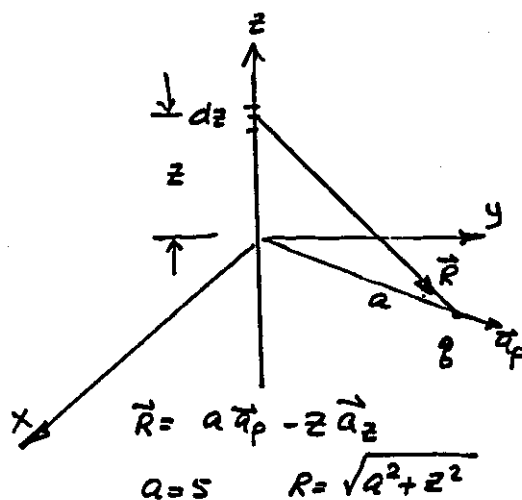
Problem 3.3

$$\vec{F} = \frac{qP}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{(q\vec{a}_p - z\vec{a}_z) dz}{(a^2 + z^2)^{3/2}}$$

$$= \frac{Pq}{2\pi\epsilon_0 a}$$

$$= \frac{500 \times 10^{-9} \times 100 \times 10^{-9} \times 36\pi \times 10^9}{2\pi \times 5}$$

$$= 180 \mu\text{N}$$



Problem 3.4 $\vec{E} = \frac{P}{2\pi\epsilon_0 d} \vec{a}_p \Rightarrow \vec{F} = -\frac{P^2}{2\pi\epsilon_0 d} \vec{a}_p$

$$\vec{F} = -\frac{(100 \times 10^{-9})^2 \times 36\pi \times 10^9}{2\pi \times 1 \times 10^{-3}} \vec{a}_p = -0.18\vec{a}_p$$

This is a force of attraction.

Problem 3.5

$$|F_e| = \left| \frac{(-1.6 \times 10^{-19})(1.6 \times 10^{-19}) 9 \times 10^9}{(0.05 \times 10^{-9})^2} \right| = 92.16 \times 10^9 \text{ N}$$

$$|F_g| = \frac{GmM}{r^2} = \frac{6.67 \times 10^{-11} \times 9.1 \times 10^{-31} \times 1.672 \times 10^{-27}}{(0.05 \times 10^{-9})^2} = 40.59 \times 10^{-48} \text{ N}$$

$$\frac{|F_e|}{|F_g|} = 2.27 \times 10^{39}$$

Electric force predominates.

Problem 3.6

$$|F_e| = \left| \frac{-1.6 \times 10^{-19} \times 1.6 \times 10^{-19} \times 9 \times 10^9}{(0.05 \times 10^{-9})^2} \right| = 92.16 \times 10^9 \text{ N}$$

Since $|F_e| = \frac{mv^2}{r}$ $v = r\omega \Rightarrow \omega = \sqrt{\frac{F}{mr}} = \sqrt{\frac{92.16 \times 10^9}{9.1 \times 10^{-31} \times 0.05 \times 10^{-9}}} = 45 \times 10^{15} \text{ rad/s}$

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} = 139.6 \times 10^{-18} \text{ s}$$

Problem 3.7

$$F = \frac{Q^2}{4\pi\epsilon_0 4L^2 \sin^2 \theta} \quad (1)$$

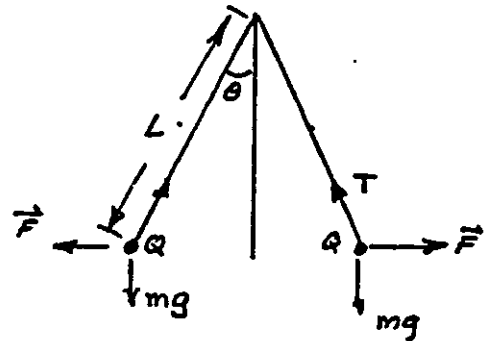
$$F = T \sin \theta \quad mg = T \cos \theta$$

Thus, $F = mg \tan \theta \quad (2)$

From (1) and (2)

$$\frac{\sin^3 \theta}{\cos^3 \theta} = \frac{Q^2}{16\pi\epsilon_0 L^2 mg}$$

$$\text{or } \frac{\tan^3 \theta}{1 + \tan^2 \theta} = \frac{Q^2}{16\pi\epsilon_0 L^2 mg}$$

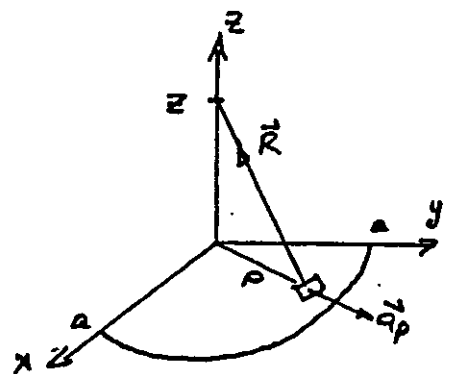


Problem 3.8

$$\vec{E} = \frac{k}{4\pi\epsilon_0} \int_0^a \int_0^{\pi/2} \frac{\cos \phi \, p \, dp \, d\phi [z \vec{a}_z - p \vec{a}_p]}{(p^2 + z^2)^{3/2}}$$

$$\vec{a}_p = \vec{a}_x \cos \phi + \vec{a}_y \sin \phi \text{ and } z = h$$

$$E_x = -\frac{k}{4\pi\epsilon_0} \int_0^a \frac{p^2 dp}{(p^2 + h^2)^{3/2}} \int_0^{\pi/2} \cos^2 \phi \, d\phi$$



$$\vec{R} = z \vec{a}_z - p \vec{a}_p$$

$$R = \sqrt{p^2 + z^2}$$

$$E_y = -\frac{k}{4\pi\epsilon_0} \int_0^a \frac{\rho^2 d\rho}{(\rho^2 + h^2)^{3/2}} \int_0^{\pi/2} \cos\phi \sin\phi d\phi$$

$$E_z = \frac{kh}{4\pi\epsilon_0} \int_0^a \frac{\rho d\rho}{(\rho^2 + h^2)^{3/2}} \int_0^{\pi/2} \cos\phi d\phi$$

Evaluating the integrals, we get

$$E_x = \frac{k}{4\pi\epsilon_0} \left[\frac{a}{\sqrt{a^2 + h^2}} - \ln \left(\frac{a + \sqrt{a^2 + h^2}}{h} \right) \right]$$

$$E_y = \frac{k}{8\pi\epsilon_0} \left[\frac{a}{\sqrt{a^2 + h^2}} - \ln \left(\frac{a + \sqrt{a^2 + h^2}}{h} \right) \right]$$

$$E_z = \frac{kh}{4\pi\epsilon_0} \left[\frac{1}{h} - \frac{1}{\sqrt{a^2 + h^2}} \right]$$

$$\int_0^{\pi/2} \cos\phi d\phi = 1$$

$$\int_0^{\pi/2} \sin\phi \cos\phi d\phi = \frac{1}{2}$$

$$\int_0^{\pi/2} \cos^2\phi d\phi = \frac{\pi}{4}$$

$$\int_0^a \frac{\rho d\rho}{(\rho^2 + h^2)^{3/2}} = -\frac{1}{\sqrt{\rho^2 + h^2}} \Big|_0^a$$

$$= -\frac{1}{\sqrt{a^2 + h^2}} + \frac{1}{h}$$

$$\int_0^a \frac{\rho^2 d\rho}{(\rho^2 + h^2)^{3/2}} = \left[-\frac{\rho}{\sqrt{\rho^2 + h^2}} + \ln(\rho + \sqrt{\rho^2 + h^2}) \right]_0^a$$

$$= -\frac{a}{\sqrt{a^2 + h^2}} + \ln \frac{a + \sqrt{a^2 + h^2}}{h}$$

Problem 3.9

$$\vec{E} = \frac{kb}{4\pi\epsilon_0} \int_0^\pi \frac{\sin\phi [h\vec{a}_z - b\vec{a}_x \cos\phi - b\vec{a}_y \sin\phi] d\phi}{(b^2 + h^2)^{3/2}}$$

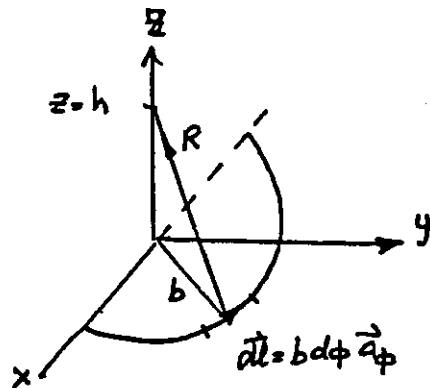
$$E_x = \frac{bk}{4\pi\epsilon_0 (b^2 + h^2)^{3/2}} \int_0^\pi -\sin\phi \cos\phi d\phi = 0$$

$$E_y = -\frac{kb^2}{4\pi\epsilon_0 (b^2 + h^2)^{3/2}} \int_0^\pi \sin^2\phi d\phi$$

$$E_z = \frac{bkh}{4\pi\epsilon_0 (b^2 + h^2)^{3/2}} \int_0^\pi \sin\phi d\phi$$

Evaluate integral 3.

$$\vec{E} = \frac{bk}{4\pi\epsilon_0 (b^2 + h^2)^{3/2}} \left[-\frac{1}{2}\pi\vec{a}_y + \pi h\vec{a}_z \right]$$



$$\vec{R} = h\vec{a}_z - b\vec{a}_\rho$$

$$R = \sqrt{h^2 + b^2}$$

$$\text{Since } \vec{a}_\rho = \vec{a}_x \cos\phi + \vec{a}_y \sin\phi$$

$$\vec{R} = (-\vec{a}_x \cos\phi + \vec{a}_y \sin\phi)b + h\vec{a}_z$$

$$\int_0^\pi \sin\phi d\phi = 2$$

$$\int_0^\pi \sin^2\phi d\phi = \frac{\pi}{2}$$

Problem 3.10 The charge distribution ρ_a yields the same \vec{E} -field as was obtained in Problem 3.8. Follow the same method to obtain \vec{E} -field due to ρ_b . The x - and y -components will be exactly the same but the z -component will be in opposite direction.

Thus,

$$E_x = \frac{A}{8\epsilon_0} \left[\frac{b}{\sqrt{b^2+h^2}} - \ln \left(\frac{b+\sqrt{b^2+h^2}}{h} \right) \right]$$

$$E_y = \frac{A}{4\pi\epsilon_0} \left[\frac{b}{\sqrt{b^2+h^2}} - \ln \left(\frac{b+\sqrt{b^2+h^2}}{h} \right) \right], \quad E_z = 0$$

Problem 3.11

$$\vec{R} = -x \vec{a}_x - y \vec{a}_y + z \vec{a}_z$$

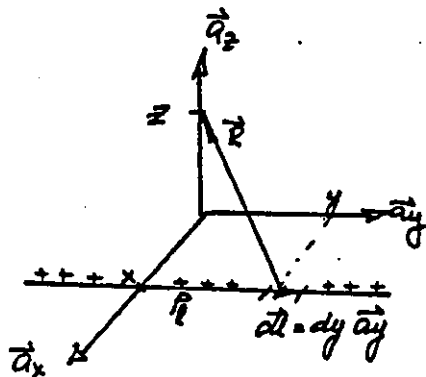
$$\vec{E} = \frac{\rho_l}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{(-x \vec{a}_x - y \vec{a}_y + z \vec{a}_z) dy}{(x^2 + y^2 + z^2)^{3/2}}$$

$$E_x = -\frac{\rho_l x}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{dy}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= -\frac{\rho_l x}{4\pi\epsilon_0} \left[\frac{1}{x^2 + z^2} \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right]_{-L/2}^{L/2} = -\frac{\rho_l x L}{4\pi\epsilon_0 (x^2 + z^2)} \cdot \frac{1}{\sqrt{x^2 + z^2 + L^2/4}}$$

$E_y = 0$

$$E_z = \frac{\rho_l z L}{4\pi\epsilon_0 (x^2 + z^2)} \cdot \frac{1}{\sqrt{x^2 + z^2 + L^2/4}}$$



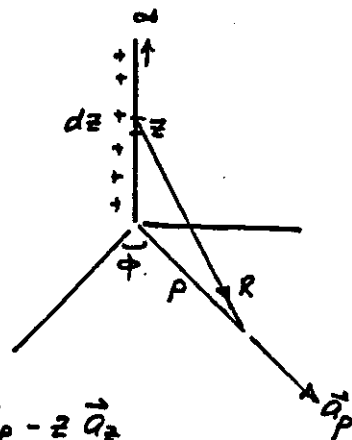
Problem 3.12

$$\vec{E} = \frac{\rho_l}{4\pi\epsilon_0} \int_0^\infty \frac{(P \vec{a}_P - z \vec{a}_z) dz}{(P^2 + z^2)^{3/2}}$$

$$E_P = \frac{\rho_l P}{4\pi\epsilon_0} \int_0^\infty \frac{dz}{(P^2 + z^2)^{3/2}} = \frac{\rho_l}{4\pi\epsilon_0 P}$$

$$E_z = -\frac{\rho_l}{4\pi\epsilon_0} \int_0^\infty \frac{z dz}{(P^2 + z^2)^{3/2}} = -\frac{\rho_l}{4\pi\epsilon_0 P}$$

$$\vec{E} = \frac{\rho_l}{4\pi\epsilon_0 P} [\vec{a}_P - \vec{a}_z]$$



$$\vec{R} = P \vec{a}_P - z \vec{a}_z$$

$$\int_0^\infty \frac{z dz}{(P^2 + z^2)^{3/2}} = -\frac{1}{\sqrt{P^2 + z^2}} \Big|_0^\infty = \frac{1}{P}$$

$$\int_0^\infty \frac{dz}{(P^2 + z^2)^{3/2}} = \frac{z}{\sqrt{P^2 + z^2}} \Big|_0^\infty = 1$$

Problem 3.10 The charge distribution ρ_a yields the same \vec{E} -field as was obtained in Problem 3.8. Follow the same method to obtain \vec{E} -field due to ρ_b . The x - and y -components will be exactly the same but the z -component will be in opposite direction.

Thus,

$$E_x = \frac{A}{8\epsilon_0} \left[\frac{b}{\sqrt{b^2+h^2}} - \ln \left(\frac{b+\sqrt{b^2+h^2}}{h} \right) \right]$$

$$E_y = \frac{A}{4\pi\epsilon_0} \left[\frac{b}{\sqrt{b^2+h^2}} - \ln \left(\frac{b+\sqrt{b^2+h^2}}{h} \right) \right], \quad E_z = 0$$

Problem 3.11

$$\vec{R} = -x \vec{a}_x - y \vec{a}_y + z \vec{a}_z$$

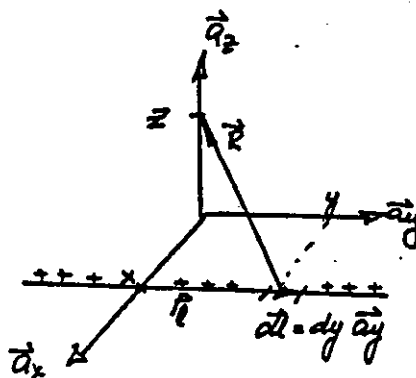
$$\vec{E} = \frac{\rho_l}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{(-x \vec{a}_x - y \vec{a}_y + z \vec{a}_z) dy}{(x^2 + y^2 + z^2)^{3/2}}$$

$$E_x = -\frac{\rho_l x}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{dy}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= -\frac{\rho_l x}{4\pi\epsilon_0} \left[\frac{1}{x^2 + z^2} \frac{y}{\sqrt{x^2 + y^2 + z^2}} \right]_{-L/2}^{L/2} = -\frac{\rho_l x L}{4\pi\epsilon_0 (x^2 + z^2)} \cdot \frac{1}{\sqrt{x^2 + z^2 + L^2/4}}$$

$$E_y = 0$$

$$E_z = \frac{\rho_l z L}{4\pi\epsilon_0 (x^2 + z^2)} \cdot \frac{1}{\sqrt{x^2 + z^2 + L^2/4}}$$



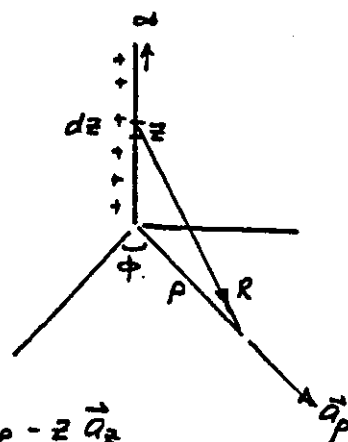
Problem 3.12

$$\vec{E} = \frac{\rho_l}{4\pi\epsilon_0} \int_0^\infty \frac{(P \vec{a}_P - z \vec{a}_z) dz}{(P^2 + z^2)^{3/2}}$$

$$E_P = \frac{\rho_l P}{4\pi\epsilon_0} \int_0^\infty \frac{dz}{(P^2 + z^2)^{3/2}} = \frac{\rho_l}{4\pi\epsilon_0 P}$$

$$E_z = -\frac{\rho_l}{4\pi\epsilon_0} \int_0^\infty \frac{z dz}{(P^2 + z^2)^{3/2}} = -\frac{\rho_l}{4\pi\epsilon_0 P}$$

$$\vec{E} = \frac{\rho_l}{4\pi\epsilon_0 P} [\vec{a}_P - \vec{a}_z]$$



$$\vec{R} = P \vec{a}_P - z \vec{a}_z$$

$$\int_0^\infty \frac{z dz}{(P^2 + z^2)^{3/2}} = -\frac{1}{\sqrt{P^2 + z^2}} \Big|_0^\infty = \frac{1}{P}$$

$$\int_0^\infty \frac{dz}{(P^2 + z^2)^{3/2}} = \frac{z}{\sqrt{P^2 + z^2}} \Big|_0^\infty = 1$$

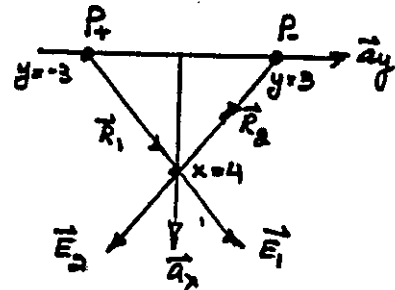
Problem 3.13 $P_+ = 1 \mu\text{C/m}$, $P_- = -1 \mu\text{C/m}$

$$\vec{R}_1 = 4\vec{a}_x + 3\vec{a}_y, R_1 = 5$$

$$\vec{R}_2 = 4\vec{a}_x - 3\vec{a}_y, R_2 = 5$$

$$\vec{E}_1 = \frac{P_+}{2\pi\epsilon_0 R_1} \vec{a}_{R_1}, \quad \vec{E}_2 = \frac{P_-}{2\pi\epsilon_0 R_2} \vec{a}_{R_2}$$

$$\begin{aligned} \vec{E} = \vec{E}_1 + \vec{E}_2 &= \frac{1 \times 10^{-6}}{50\pi\epsilon_0} [4\vec{a}_x + 3\vec{a}_y - 4\vec{a}_x + 3\vec{a}_y] \\ &= 4320 \vec{a}_y \text{ V/m} \end{aligned}$$



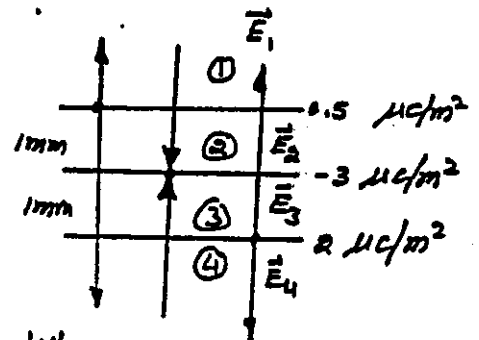
Problem 3.14

$$\begin{aligned} \text{Region-1: } \vec{E}_1 &= \frac{1}{2\epsilon_0} [0.5 - 3 + 2] 10^{-6} \vec{a}_z \\ &= -28.274 \vec{a}_z \text{ kV/m} \end{aligned}$$

$$\text{Region-2: } \vec{E}_2 = \frac{1}{2\epsilon_0} [-0.5 - 3 + 2] 10^{-6} \vec{a}_z = -84.823 \vec{a}_z \text{ kV/m}$$

$$\text{Region-3: } \vec{E}_3 = \frac{1}{2\epsilon_0} [-0.5 + 3 + 2] 10^{-6} \vec{a}_z = 254.47 \vec{a}_z \text{ kV/m}$$

$$\text{Region-4: } \vec{E}_4 = \frac{1}{2\epsilon_0} [-0.5 + 3 - 2] 10^{-6} \vec{a}_z = 28.274 \vec{a}_z \text{ kV/m}$$



Problem 3.15 $\vec{R} = z\vec{a}_z - b\vec{a}_\rho$ $P = P_0 \sin\phi \cos\phi$
 $P_0 = 1.2 \times 10^{-6}$
 $= z\vec{a}_z - b \cos\phi \vec{a}_x - b \sin\phi \vec{a}_y$

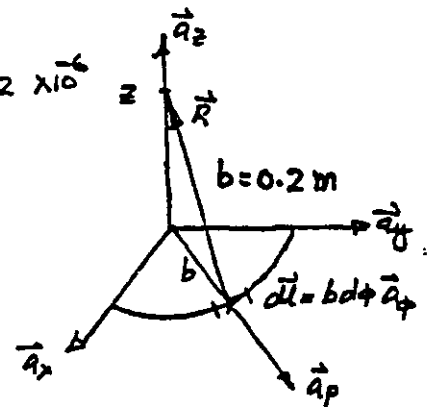
$$\begin{aligned} \vec{E} &= \frac{P_0 b}{4\pi\epsilon_0} \left[\int_0^{\pi/2} -b \sin\phi \cos^2\phi d\phi \vec{a}_x \right. \\ &\quad - \int_0^{\pi/2} b \sin^2\phi \cos\phi d\phi \vec{a}_y \\ &\quad \left. + \int_0^{\pi/2} z \sin\phi \cos\phi d\phi \vec{a}_z \right] \frac{1}{(b^2 + z^2)^{3/2}} \end{aligned}$$

$$= \frac{P_0 b}{4\pi\epsilon_0 (b^2 + z^2)^{3/2}} \left[\frac{b}{3} \cos^3\phi \vec{a}_x - \frac{b}{3} \sin^3\phi \vec{a}_y + \frac{z}{2} \sin^2\phi \vec{a}_z \right]_0^{\pi/2}$$

$$= \frac{P_0 b}{4\pi\epsilon_0 (b^2 + z^2)^{3/2}} \left[-\frac{1}{3} \vec{a}_x - \frac{1}{3} \vec{a}_y + \frac{z}{2} \vec{a}_z \right]. \text{ Now substitute the values}$$

$$\text{at } z=1 \quad \vec{E} = -135.77 \vec{a}_x - 135.77 \vec{a}_y + 1018.3 \vec{a}_z \text{ V/m}$$

$$\text{at } z=0 \quad \vec{E} = -18000 (\vec{a}_x + \vec{a}_y) \text{ V/m}$$



Problem 3.16 $\vec{R} = a\vec{a}_\rho - z\vec{a}_z$ $R = \sqrt{a^2 + z^2}$

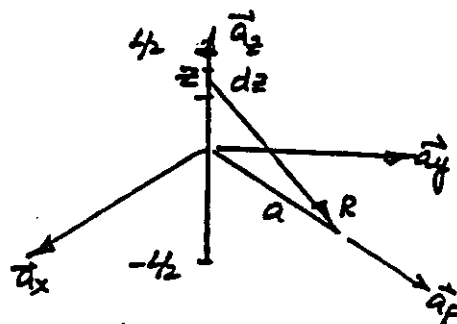
$$\vec{E} = \frac{\rho_0}{4\pi\epsilon_0} \int_{-L/2}^{L/2} \frac{z dz}{(a^2 + z^2)^{3/2}} [a\vec{a}_\rho - z\vec{a}_z]$$

$$= \vec{a}_\rho \frac{\rho_0 a}{4\pi\epsilon_0} \left[-\frac{1}{\sqrt{z^2 + a^2}} \right]_{-L/2}^{L/2} - \vec{a}_z \frac{\rho_0}{4\pi\epsilon_0} \left[-\frac{z}{\sqrt{a^2 + z^2}} + \ln \left[z + \sqrt{a^2 + z^2} \right] \right]_{-L/2}^{L/2}$$

$$= \vec{a}_\rho \frac{\rho_0 a}{4\pi\epsilon_0} \left[-\frac{1}{\sqrt{a^2 + L^2/4}} + \frac{1}{\sqrt{a^2 + L^2/4}} \right] - \vec{a}_z \frac{\rho_0}{4\pi\epsilon_0} \left[-\frac{L}{\sqrt{a^2 + L^2/4}} + \ln \left[\frac{L}{2} + \sqrt{a^2 + \frac{L^2}{4}} \right] - \left(-\frac{L}{2} + \sqrt{a^2 + \frac{L^2}{4}} \right) \right]$$

$$E_z = -\frac{\rho_0}{4\pi\epsilon_0} \left[-\frac{L}{\sqrt{a^2 + L^2/4}} + \ln \left[\frac{L}{2} + \sqrt{a^2 + \frac{L^2}{4}} \right] - \left(-\frac{L}{2} + \sqrt{a^2 + \frac{L^2}{4}} \right) \right]$$

= -2397.51 V/m when values are substituted.



$$\frac{L}{2} = 10 \text{ m}$$

$$a = 2 \text{ m}$$

$$\rho_0 = 100 \text{ nC/m}$$

Problem 3.17 $\vec{R} = -b\vec{a}_\rho + (h-z)\vec{a}_z$

$$R = \sqrt{b^2 + (h-z)^2}$$

$$\vec{E} = \frac{\rho_s}{4\pi\epsilon_0} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^{2\pi} \frac{b d\phi dz [-b\vec{a}_\rho - (z-h)\vec{a}_z]}{[b^2 + (z-h)^2]^{3/2}}$$

$$\vec{a}_\rho = \vec{a}_x \cos \phi + \vec{a}_y \sin \phi \text{ and}$$

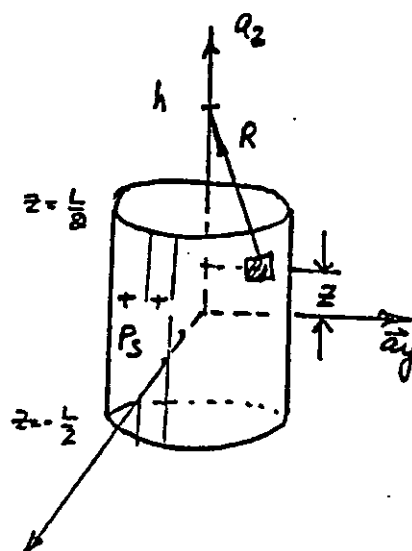
$$\int_0^{2\pi} \cos \phi d\phi = 0, \quad \int_0^{2\pi} \sin \phi d\phi = 0$$

Thus, $E_x = 0, E_y = 0$

$$E_z = -\frac{2\pi b \rho_s}{4\pi\epsilon_0} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{(z-h) dz}{[b^2 + (z-h)^2]^{3/2}}$$

$$= \frac{2\pi b \rho_s}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{b^2 + (z-h)^2}} \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= \frac{b \rho_s}{2\epsilon_0} \left[\frac{1}{\sqrt{b^2 + (\frac{L}{2} - h)^2}} - \frac{1}{\sqrt{b^2 + (\frac{L}{2} + h)^2}} \right]$$



when $h=0, E_z=0$

when $h=L/2,$

$$E_z = \frac{b \rho_s}{2\epsilon_0} \left[\frac{1}{b} - \frac{1}{\sqrt{b^2 + L^2}} \right]$$

when $h=-L/2,$

$$E_z = \frac{b \rho_s}{2\epsilon_0} \left[\frac{1}{\sqrt{b^2 + L^2}} - \frac{1}{b} \right]$$

Problem 3.18 $\vec{D} = 6y\vec{a}_x + 2x\vec{a}_y + 14xy\vec{a}_z$ mC/m^2

a) $d\vec{s} = dydz\vec{a}_x + dx dz\vec{a}_y$

$$\int_V \vec{D} \cdot d\vec{s} = \int_0^2 6y dy \int_0^2 dz + \int_0^2 2x dx \int_0^2 dz$$

$$= 32 \text{ mC}$$

b) $d\vec{s} = \rho d\rho d\phi \vec{a}_z$

$$\int_V \vec{D} \cdot d\vec{s} = \int_0^1 \int_0^{2\pi} 14xy \rho d\rho d\phi \quad \begin{matrix} x = \rho \cos\phi \\ y = \rho \sin\phi \end{matrix}$$

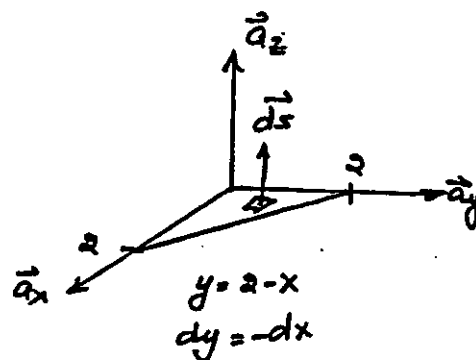
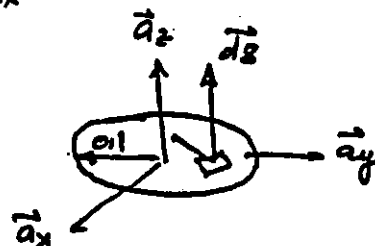
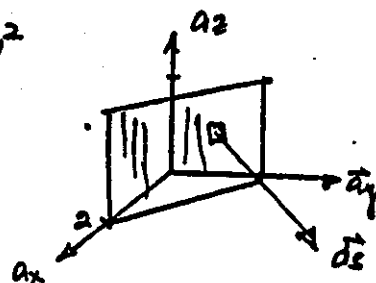
$$= \int_0^1 \rho^3 d\rho \int_0^{2\pi} \sin\phi \cos\phi d\phi = 0$$

c) $d\vec{s} = dx dy \vec{a}_z$

$$\int_V \vec{D} \cdot d\vec{s} = \int_0^2 14x dx \int_{2-x}^{2-x} y dy$$

$$= 7 \int_0^2 x [(2-x)^2 - 4] dx =$$

$$= \left[\frac{7x^4}{4} - \frac{28x^3}{3} \right]_0^2 = -46.67 \text{ mC}$$



Problem 3.19 $\oint_V \vec{D} \cdot d\vec{s} = Q_{enc} \Rightarrow D_p = \frac{a P_s}{\rho}$

$$a = 0.2 \text{ m}$$

$$P_s = 10 \text{ mC/m}^2$$

$$\psi = \int_V \vec{D} \cdot d\vec{s} = \int_{\pi/4}^{3\pi/4} \frac{a P_s}{\rho} \rho d\phi \int_2^4 dz = a \pi P_s = 6.28 \text{ mC}$$

Problem 3.20 $\rho < b: \vec{E} = 0 \quad \rho > b: \vec{E} = \frac{b P_s}{\epsilon_0 \rho} \vec{a}_\rho \quad D_p = \frac{b P_s}{\rho}$

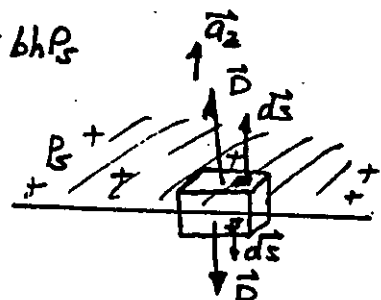
$$\psi = \int_V \vec{D} \cdot d\vec{s} = b P_s \int_0^{\pi/2} \frac{1}{\rho} \rho d\phi \int_0^h dz = \frac{\pi}{2} b h P_s$$

Problem 3.21

$$\oint_V \vec{D} \cdot d\vec{s} = Q_{enc} \Rightarrow 2 D_z A = P_s A$$

$$D_z = \frac{P_s}{2}$$

$$E_z = \frac{P_s}{2 \epsilon_0}$$



A = Area of Top and bottom surfaces.

Problem 3.22 $\vec{E} = \frac{Q_{enc}}{4\pi\epsilon_0 r^2} \vec{a}_r$

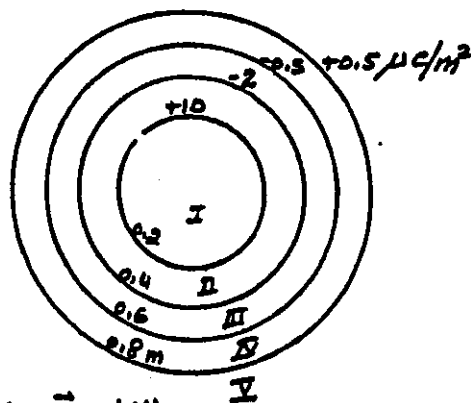
$\vec{E}|_{r=0.1} = 0 \quad \because Q_{enc} = 0$

$\vec{E}|_{r=0.3} = \frac{4\pi(0.2)^2 \cdot 10 \times 10^{-6}}{(0.3)^2} 9 \times 10^9 \vec{a}_r$
 $= 502.65 \vec{a}_r \text{ kV/m}$

$\vec{E}|_{r=0.5} = 4\pi \frac{[0.2^2 \times 10 - 0.4^2 \times 2] 10^{-6}}{(0.5)^2} 9 \times 10^9 \vec{a}_r = 36.19 \vec{a}_r \text{ kV/m}$

$\vec{E}|_{r=0.7} = 4\pi \frac{[0.2^2 \times 10 - 0.4^2 \times 2 - 0.6^2 \times 0.5] 10^{-6}}{(0.7)^2} 9 \times 10^9 \vec{a}_r = -23.08 \vec{a}_r \text{ kV/m}$

$\vec{E}|_{r=1} = 4\pi \frac{[0.2^2 \times 10 - 0.4^2 \times 2 - 0.6^2 \times 0.5 + 0.8^2 \times 0.5] 10^{-6}}{(1)^2} 9 \times 10^9 \vec{a}_r = 24.88 \vec{a}_r \text{ kV/m}$



Problem 3.23 $\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r \Rightarrow \psi = \int \vec{D} \cdot d\vec{s} = \frac{Q}{4\pi} \int_0^\theta \frac{1}{r^2} r^2 \sin\theta d\theta \int_0^{2\pi} d\phi$
 $= \frac{Q}{2} (1 - \cos\theta_0)$

Problem 3.24 $r < a, D = 0, E = 0$

$a \leq r \leq b: Q_{enc} = k(P-a)2\pi h \Rightarrow 2\pi r D_p = k(P-a)2\pi h \Rightarrow D_p = \frac{k(P-a)}{r}, E_p = \frac{k}{\epsilon_0} (1 - \frac{a}{r})$

$b \leq r \leq c: D = 0, E = 0$

$r > c: D_p 2\pi r h = k(b-a)2\pi h \Rightarrow D_p = \frac{k(b-a)}{r}, E_p = \frac{k(b-a)}{\epsilon_0 r}$

Problem 3.25 $r < b: Q_{enc} = k \int_0^r dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = 4\pi r k$

$\oint \vec{D} \cdot d\vec{s} = 4\pi r^2 D_r \Rightarrow D_r = \frac{k}{r}$

$r > b: Q_{enc} = 4\pi b k \Rightarrow D_r = \frac{k b}{r^2}$

Problem 3.26

$Q_{enc} = 2\pi a L P_s$

$a = 0.05 \text{ m}$

$\oint \vec{D} \cdot d\vec{s} = 2\pi P L Q_p \Rightarrow D_p = \frac{a P_s}{r}$

$E_p = \frac{a P_s}{\epsilon_0 r}$ at $r = 1 \text{ m}, E_p = 100 \times 10^3 \text{ V/m} \Rightarrow$

$P_s = \frac{100 \times 10^3 \times 10^{-9}}{36\pi \cdot 0.05} = 17.68 \mu\text{C/m}^2$

Problem 3.27 $\vec{R} = \rho \vec{a}_\rho - z \vec{a}_z$ $R = \sqrt{\rho^2 + z^2}$ where $\rho = a$

$$V = \int_0^L \frac{\rho_0 dz}{4\pi\epsilon_0 \sqrt{\rho^2 + z^2}} = \frac{\rho_0}{4\pi\epsilon_0} \ln[z + \sqrt{z^2 + \rho^2}] \Big|_0^L$$

$$= \frac{\rho_0}{4\pi\epsilon_0} \ln\left[\frac{L + \sqrt{L^2 + \rho^2}}{\rho}\right] \quad \text{at } \rho = a, V(a) = \frac{\rho_0}{4\pi\epsilon_0} \ln\left[\frac{L + \sqrt{L^2 + a^2}}{a}\right]$$

$$\vec{E} = -\nabla V = \frac{\rho_0}{4\pi\epsilon_0} \left[\frac{L/\rho}{\sqrt{L^2 + \rho^2}} \right] \vec{a}_\rho \quad \text{at } \rho = a, \vec{E}(a) = \frac{\rho_0}{4\pi\epsilon_0 a} \left[\frac{L}{\sqrt{L^2 + a^2}} \right]$$

Problem 3.28

$$W = \frac{Q_1 Q_2}{4\pi\epsilon_0 R} = \frac{500 \times 10^{-9} \times (-600) \times 10^{-9}}{1 \times 10^{-3}} \cdot 9 \times 10^9 = -2.7 \text{ J}$$

The electric field due to 500 nC is attracting the charge of -600 nC towards it. Thus, energy released = 2.7 J

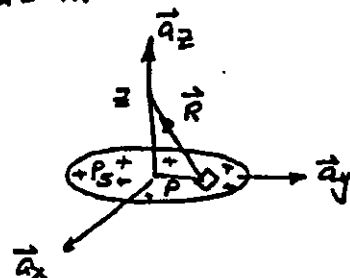
Problem 3.29

$$V = \frac{\rho_s}{4\pi\epsilon_0} \int_0^b \int_0^{2\pi} \frac{\rho d\phi d\rho}{\sqrt{\rho^2 + z^2}}$$

or

$$V = \frac{\rho_s}{2\epsilon_0} \int_0^b \frac{\rho d\rho}{\sqrt{\rho^2 + z^2}} = \frac{\rho_s}{2\epsilon_0} \left[\sqrt{\rho^2 + z^2} \right]_0^b$$

$$= \frac{\rho_s}{2\epsilon_0} \left[\sqrt{b^2 + z^2} - z \right]$$



$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial z} \vec{a}_z = -\frac{\rho_s}{2\epsilon_0} \left[\frac{z}{\sqrt{b^2 + z^2}} - 1 \right] \vec{a}_z$$

Problem 3.30

From Problem 3.29, by changing the lower limit to a , we get

$$V = \frac{\rho_s}{2\epsilon_0} \left[\sqrt{b^2 + z^2} - \sqrt{a^2 + z^2} \right]$$

and

$$\vec{E} = \frac{\rho_s z}{2\epsilon_0} \left[\frac{1}{\sqrt{a^2 + z^2}} - \frac{1}{\sqrt{b^2 + z^2}} \right] \vec{a}_z$$

Problem 3.31

$$V = \frac{Q}{4\pi\epsilon_0 R} \Rightarrow Q = 4\pi\epsilon_0 R V$$

When $V = 9 \text{ kV}$ at $R = 0.2 \text{ m}$, $Q = \frac{0.2 \times 9 \times 10^3}{9 \times 10^9} = 200 \text{ nC}$

(a) When $V = 18 \text{ kV}$, $R = \frac{200 \times 10^{-9} \times 9 \times 10^9}{18 \times 10^3} = 0.1 \text{ m or } 10 \text{ cm}$

(b) When $V = 3 \text{ kV}$, $R = 0.6 \text{ m or } 60 \text{ cm}$.

Problem 3.32 $\vec{E} = 10\vec{a}_x + 20\vec{a}_y + 20\vec{a}_z$ kV/m

From (0,0,0) to (3,0,0): $W_1 = -q \int_C \vec{E} \cdot d\vec{l} = -0.1 \times 10^{-9} \times 10 \times 10^3 \times 3 = -3 \mu J$

From (3,0,0) to (3,4,0): $W_2 = -0.1 \times 10^{-9} \times 20 \times 10^3 \times 4 = -8 \mu J$

Total work done: $W = -11 \mu J$

Directly from (0,0,0) to (3,4,0): $dW = -q \vec{E} \cdot d\vec{l} = -q E_x dx - q E_y dy$

$$W = -0.1 \times 10^{-9} \int_0^3 10 \times 10^3 dx - 0.1 \times 10^{-9} \int_0^4 20 \times 10^3 dy = -11 \mu J$$

Problem 3.33 $\vec{E} = 10 \times 10^3 \vec{a}_x$ V/m $V=0$ at $x=0$, at any point $P(x,y,z)$

$$V_P = - \int_0^x \vec{E} \cdot d\vec{l} = -10 \times 10^3 x. \text{ Thus, } V_P(x,y,z) = -10x \text{ kV}$$

Problem 3.34 $V = 10x^2 + 20y^2 + 5z$ V :

$$\vec{E} = -\nabla V = -[20x \vec{a}_x + 40y \vec{a}_y + 5] \text{ V/m} \quad \nabla \times \vec{E} = 0$$

This potential function can exist because $\nabla \times \vec{E} = 0$

Problem 3.35 : Inside: $Q_{enc} = \int_0^P \rho_P d\rho \int_0^{2\pi} d\phi \int_0^h dz = \pi P^2 h \rho_P$

$$\oint \vec{E} \cdot d\vec{s} = 2\pi P h D_P \Rightarrow D_P = \frac{P \rho_P}{2} \Rightarrow E_P = \frac{P \rho_P}{2\epsilon_0} \quad P \leq a$$

outside: $Q_{enc} = \pi a^2 h \rho_P$ and $E_P = \frac{a^2 \rho_P}{2 P \epsilon_0} \quad P \geq a$

Potential: outside: Let $V=0$ when $P=b$, then

$$V(P) = - \int_b^P \frac{a^2 \rho_P}{2\epsilon_0} \frac{1}{P} dP = \frac{\rho_P a^2}{2\epsilon_0} \ln(b/P)$$

when $P=a$: $V(a) = \frac{\rho_P a^2}{2\epsilon_0} \ln(b/a)$

Potential distribution inside $P < a$:

$$V(P) - V(a) = - \int_a^P \frac{\rho_P}{2\epsilon_0} P dP = + \frac{\rho_P}{4\epsilon_0} [a^2 - P^2]$$

Thus, $V(P) = \frac{\rho_P}{4\epsilon_0} [a^2 - P^2] + \frac{\rho_P a^2}{2\epsilon_0} \ln(b/a)$

Problem 3.36 $\vec{E} = 10y \vec{a}_x + 10x \vec{a}_y + 2z \vec{a}_z$ kV/m

$$\vec{E} \cdot d\vec{l} = -20y d\phi \vec{a}_x \cdot \vec{a}_\phi + 20x d\phi \vec{a}_y \cdot \vec{a}_\phi \quad \text{kV}$$

$$\vec{a}_x \cdot \vec{a}_\phi = -\sin\phi, \quad \vec{a}_y \cdot \vec{a}_\phi = \cos\phi, \quad x = 2\cos\phi, \quad y = 2\sin\phi$$

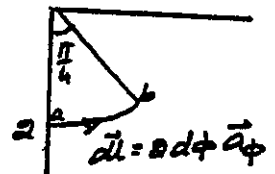
$$\vec{E} \cdot d\vec{l} = (-40 \sin^2\phi + 40 \cos^2\phi) d\phi \quad \text{kV} = 40 \times 10^3 \cos 2\phi d\phi$$

Hence $V_{ba} = -\int_a^b \vec{E} \cdot d\vec{l} = -40 \times 10^3 \int_0^{\pi/4} \cos 2\phi d\phi = -140 \text{ kV}$

$$W_{ba} = q V_{ba} = -0.5 \times 10^{-3} \times 40 \times 10^3 = -20 \text{ J}$$

\vec{E} field is doing the work.

$\nabla \times \vec{E} = 0$
 \vec{E} can exist.



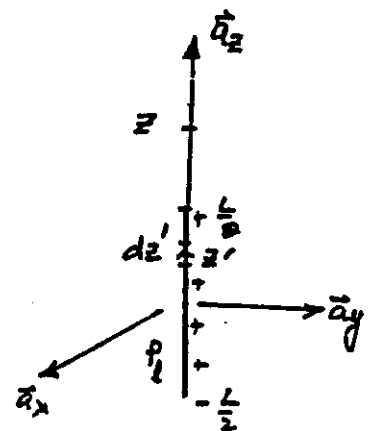
Problem 3.37

$$V = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\rho_l}{4\pi\epsilon_0} \frac{dz'}{z-z'} = -\frac{\rho_l}{4\pi\epsilon_0} \left[\ln(z-z') \right]_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$= -\frac{\rho_l}{4\pi\epsilon_0} \left[\ln\left(\frac{z-L/2}{z+L/2}\right) \right]$$

$$\vec{E} = -\nabla V = \frac{\rho_l}{4\pi\epsilon_0} \left[\frac{1}{z-L/2} - \frac{1}{z+L/2} \right] \vec{a}_z$$

$$= \frac{\rho_l}{4\pi\epsilon_0} \left[\frac{L}{z^2 - L^2/4} \right] \vec{a}_z$$



Problem 3.38 $d = 1 \mu\text{m}$ $\vec{P} = q d \vec{a}_z = 10 \times 10^{-9} \times 1 \times 10^{-6} \vec{a}_z = 10^{-14} \vec{a}_z$ Cm.

$$V_p = \frac{\vec{P} \cdot \vec{a}_r}{4\pi\epsilon_0 r^2} \quad r=1 \text{ m}, \quad \vec{a}_r = \vec{a}_z \Rightarrow V_p = \frac{10^{-14} \times 9 \times 10^9}{1} = 90 \mu\text{V}$$

$$\vec{E}_p = \frac{P}{4\pi\epsilon_0 r^3} [2\cos\theta \vec{a}_r + \sin\theta \vec{a}_\theta] \quad \because \theta=0$$

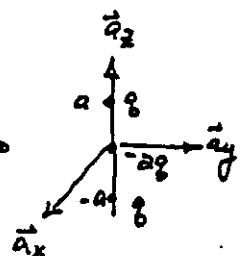
$$= 10^{-14} \times 9 \times 10^9 \times 2 \vec{a}_z = 180 \vec{a}_z \text{ V/m}$$

Problem 3.39

$$V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{z-a} - \frac{2}{z} + \frac{1}{z+a} \right] = \frac{2a^2 q}{z(z^2 - a^2) 4\pi\epsilon_0}$$

for $z \gg a$ $V \approx \frac{2a^2 q}{4\pi\epsilon_0 z^3}$

$$\vec{E} = -\nabla V = \frac{6a^2 q}{4\pi\epsilon_0 z^4} \vec{a}_z$$



Problem 3.40 $V = \frac{\vec{P} \cdot \vec{a}_r}{4\pi\epsilon_0 r^2} + \frac{q}{4\pi\epsilon_0 r^2} = \frac{1}{4\pi\epsilon_0 r^2} [q + \vec{P} \cdot \vec{a}_r]$

$$\vec{E} = \frac{qd}{4\pi\epsilon_0 r^3} [2\cos\theta \vec{a}_r + \sin\theta \vec{a}_\theta] + \frac{q\vec{a}_r}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0 r^3} [(2d\cos\theta + r)\vec{a}_r + d\sin\theta \vec{a}_\theta]$$

Problem 3.41 Inside the inner conductor: $\vec{E} = 0$ $P < a$

$a \leq P \leq b$ $E_P = \frac{Q_P}{P\epsilon_0}$ and $E_P = 0$ for $P > b$ when the outer conductor is grounded.

when the outer conductor is not grounded,

$$E_P = 0 \quad b \leq P \leq c \quad \text{and} \quad E_P = \frac{Q_P}{P\epsilon_0} \quad P > c$$

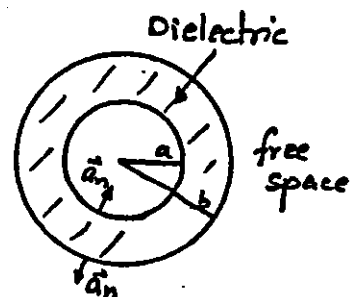
Problem 3.42 $\vec{E} = 0$ $P < a$.

$$a \leq P \leq b \quad E_P = \frac{Q_P}{P\epsilon} , \quad D_P = \frac{Q_P}{P} , \quad \omega = \frac{1}{2} \frac{a^2 P^2}{\epsilon P^2}$$

$$\vec{P} = \vec{D} - \epsilon_0 \vec{E} \Rightarrow P_P = \frac{Q_P}{P} \left(\frac{\epsilon_r - 1}{\epsilon_r} \right)$$

$$P_{tb} = -\vec{\nabla} \cdot \vec{P} = 0 \quad P_{sb}|_{P=a} = \vec{P} \cdot \vec{a}_n = -P_s \left(\frac{\epsilon_r - 1}{\epsilon_r} \right), \quad P_{sb}|_{P=b} = \frac{a}{b} P_s \left(\frac{\epsilon_r - 1}{\epsilon_r} \right)$$

$$P > b: \quad E_P = \frac{Q_P}{\epsilon_0 P} , \quad D_P = \frac{Q_P}{P} , \quad \omega = \frac{1}{2} \frac{a^2 P^2}{\epsilon_0 P^2}$$



Problem 3.43 ϵ is a function of location. i.e. $\epsilon(x, y, z)$

$$\vec{\nabla} \cdot \vec{D} = 0 \Rightarrow \vec{\nabla} \cdot \epsilon \vec{E} = 0 \Rightarrow \epsilon \vec{\nabla} \cdot \vec{E} + \vec{E} \cdot \vec{\nabla} \epsilon = 0$$

$$\text{Thus:} \quad \vec{\nabla} \cdot \vec{E} = - \frac{\vec{E} \cdot \vec{\nabla} \epsilon}{\epsilon}$$

Problem 3.44 $\epsilon = \alpha z^n$, Given $\vec{E} = E \vec{a}_z$

$$\text{From Problem 3.43,} \quad \vec{\nabla} \cdot \vec{E} = - \frac{\vec{E} \cdot \vec{\nabla} \epsilon}{\epsilon}$$

$$\text{However,} \quad \vec{\nabla} \epsilon = \frac{\partial}{\partial z} (\alpha z^n) \vec{a}_z = n\alpha z^{n-1} \vec{a}_z = \frac{\alpha n}{z} z^n \vec{a}_z$$

$$\text{and} \quad \vec{\nabla} \cdot \vec{E} = - \frac{nE}{z} = - \frac{n}{z} \epsilon \vec{a}_z$$

where E is the magnitude of the electric field intensity.

Problem 3.45 If ρ_s is the surface charge density, $Q = 4\pi a^2 \rho_s$.

From Gauss' Law: $D_r = \frac{a^2 \rho_s}{r^2}$, $\epsilon = \epsilon_0 \left[\frac{a+r}{r} \right] \Rightarrow E_r = \frac{a^2 \rho_s}{\epsilon_0 r (a+r)}$

$$P_r = D_r - \epsilon_0 E_r = \frac{a^2 \rho_s}{r^2 (a+r)}, \quad P_{sb}|_{r=a} = -\frac{\rho_s}{2}, \quad P_{sb}|_{r=b} = \frac{a^2 \rho_s}{b^2 (a+b)}$$

$$P_{rb} = -\nabla \cdot \vec{P} = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 P_r) = \frac{a^2 \rho_s}{r^3 (a+r)^2}, \quad w = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{\rho_s^2 a^4}{\epsilon_0 r^3 (a+r)}$$

$$V_r = -\int_{\infty}^r \vec{E} \cdot d\vec{l} = -\frac{a^2 \rho_s}{\epsilon_0} \int_{\infty}^r \frac{dr}{r(a+r)} = -\frac{Q}{4\pi \epsilon_0 a} \int_{\infty}^r \frac{1}{r} dr + \frac{Q}{4\pi \epsilon_0 a} \int_{\infty}^r \frac{dr}{r+a}$$

$$= \frac{Q}{4\pi \epsilon_0 a} \ln\left(\frac{a+r}{r}\right)$$

Problem 3.46 : $W = \frac{Q_1 Q_2}{4\pi \epsilon r} = \frac{10 \times 10^{-6} \times 10 \times 10^{-6} \times 9 \times 10^9}{5.5 \times 10 \times 10^{-3}} = 16.36 \text{ J}$

Problem 3.47 $E_z = -\frac{\rho_s}{\epsilon}$, $w = \frac{1}{2} \frac{\rho_s^2}{\epsilon}$ $A = 400 \times 10^{-4} \text{ m}^2$ $d = 1 \times 10^{-3} \text{ m}$

$$W = \frac{1}{2} \frac{\rho_s^2}{\epsilon} A d = \frac{1}{2} \times (250 \times 10^{-9})^2 \times \frac{36\pi \times 10^9}{2} \times 400 \times 10^{-4} \times 1 \times 10^{-3} = 70.7 \text{ nJ}$$

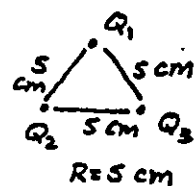
Problem 3.48 $R = \sqrt{4^2 + 3^2} = 5$

$$W = \frac{Q_1 Q_2}{4\pi \epsilon_0 R} = \frac{10 \times 10^{-9} \times 300 \times 10^{-9} \times 9 \times 10^9}{5} = 54 \text{ } \mu\text{J}$$

Problem 3.49 $V_1 = \frac{Q_2 + Q_3}{4\pi \epsilon_0 R}$, $V_2 = \frac{Q_1 + Q_3}{4\pi \epsilon_0 R}$, $V_3 = \frac{Q_1 + Q_2}{4\pi \epsilon_0 R}$

$$W = \frac{1}{2} \sum Q_i V_i = \frac{1}{4\pi \epsilon_0 R} [Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1]$$

$$= \frac{9 \times 10^{-9}}{5} [100 \times 200 + 200 \times 300 + 300 \times 100] 10^{18} = 19.8 \text{ mJ}$$



Problem 3.50

$$E_p = \frac{100}{\rho} \quad D_p = \frac{100 \epsilon}{\rho} \quad w = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{1}{2} \epsilon \left(\frac{100}{\rho} \right)^2 \quad a = 0.2 \text{ m}$$

$$b = 0.5 \text{ m}$$

$$W = \frac{\epsilon (100)^2}{2} \int_{0.2}^{0.5} \frac{1}{\rho} d\rho \int_0^{2\pi} d\phi \int_0^1 dz = \frac{\epsilon (100)^2}{2} \ln\left(\frac{0.5}{0.2}\right) 2\pi (1) = 1.4 \text{ } \mu\text{J}$$

$$\text{where } \epsilon = 5.5 \epsilon_0.$$

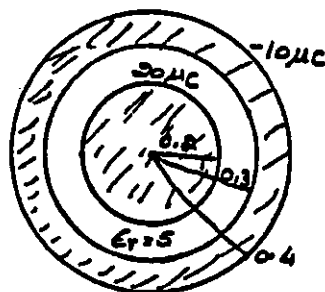
Problem 3.51 When Q is the total charge on the sphere, the potential on the surface of the sphere: $V = \frac{Q}{4\pi\epsilon_0 a}$

$$dW = V dQ \Rightarrow W = \int_0^Q \frac{Q}{4\pi\epsilon_0 a} dQ = \frac{1}{2} \frac{Q^2}{4\pi\epsilon_0 a} = \frac{1}{2} QV$$

Problem 3.52

0.25 to 0.3: $D_r = \frac{Q}{4\pi r^2}$, $E_r = \frac{Q}{4\pi\epsilon r^2}$ $\epsilon = 5\epsilon_0$

$W_1 = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{Q^2}{(4\pi)^2 \epsilon r^4}$ where $Q = 20 \times 10^{-6} \text{ C}$



$$W_1 = \frac{1}{2} \frac{Q^2}{(4\pi)^2 \epsilon} \int_{0.2}^{0.3} \frac{1}{r^2} dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{1}{2} \frac{Q^2}{16\pi^2} \cdot \frac{4\pi}{\epsilon} \left[\frac{1}{0.2} - \frac{1}{0.3} \right] = 0.6 \text{ J}$$

$r > 0.4 \text{ m}$: $Q_{enc} = 10 \mu\text{C} \Rightarrow W_2 = \frac{1}{2} \frac{Q_{enc}^2}{16\pi^2 \epsilon_0} \cdot \frac{4\pi}{\epsilon_0} \left[\frac{1}{0.4} - \frac{1}{\infty} \right] = 1.125 \text{ J}$

$$W = W_1 + W_2 = 1.725 \text{ J}$$

Problem 3.53 $\vec{E}_1 = 12\vec{a}_x + 24\vec{a}_y - 36\vec{a}_z \text{ V/m}$, $\epsilon_1 = 4\epsilon_0$, Interface at $x=5$

$E_{tan1} = E_{tan2} \Rightarrow E_{2y} = 24 \text{ V/m}$ $E_{2z} = -36 \text{ V/m}$

$D_{in} = D_{2n} \Rightarrow \epsilon_1 E_{1x} = \epsilon_2 E_{2x} \Rightarrow E_{2x} = \frac{4}{16} \cdot 12 = 3 \text{ V/m}$

$$\vec{E}_2 = 3\vec{a}_x + 24\vec{a}_y - 36\vec{a}_z \text{ V/m}$$

Problem 3.54 $E_r|_{r=20\text{cm}} = 10 \times 10^6 \text{ V/m} \Rightarrow D_r = 10 \times 10^6 \times \frac{10^{-9}}{36\pi} = 88.42 \mu\text{C/m}^2$

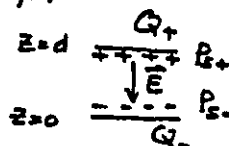
Thus, $P_s|_{r=20\text{cm}} = 88.42 \mu\text{C/m}^2$.

Problem 3.55 $E_2 = -10 \times 10^3 \text{ V/m}$ $A = 25 \times 10^{-4} \text{ m}^2$, $d = 1 \times 10^{-3} \text{ m}$

$D_2 = \epsilon E_2 = -\frac{36 \times 10^{-9}}{36\pi} \times 10 \times 10^3 = -31.83 \mu\text{C/m}^2$ $\epsilon_r = 3.6$

Thus, $P_{s+} = 318.3 \text{ nC/m}^2$

and $P_{s-} = -318.3 \text{ nC/m}^2$



$Q_+ = P_{s+} A = 7957 \text{ pC}$

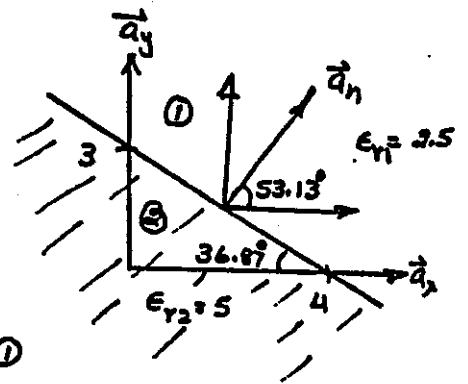
$Q_- = P_{s-} A = -795.7 \text{ pC}$

$P_C = 10^{-12} \text{ C}$

Problem 3.56 $\vec{a}_n = \cos(53.13^\circ) \vec{a}_x + \sin(53.13^\circ) \vec{a}_y$
 $= 0.6 \vec{a}_x + 0.8 \vec{a}_y$

$\vec{E}_1 = 25 \vec{a}_x + 50 \vec{a}_y + 25 \vec{a}_z \text{ V/m,}$

Let $\vec{E}_2 = E_{x2} \vec{a}_x + E_{y2} \vec{a}_y + E_{z2} \vec{a}_z$



$\vec{a}_n \cdot (\vec{D}_1 - \vec{D}_2) = 0 \Rightarrow 0.8 E_{y2} + 0.6 E_{x2} = 27.5 \quad (1)$

$\vec{a}_n \times (\vec{E}_1 - \vec{E}_2) = \vec{a}_x [25 - E_{z2}] 0.8 + \vec{a}_y [E_{z2} - 25] 0.6 + \vec{a}_z (0.8 E_{x2} - 0.6 E_{y2} + 10) = 0$

Thus, $E_{z2} = 25$ and $0.6 E_{y2} - 0.8 E_{x2} = 10 \quad (2)$

From (1) and (2): $E_{x2} = 8.5$ and $E_{y2} = 28$.

Thus, $\vec{E}_2 = 8.5 \vec{a}_x + 28 \vec{a}_y + 25 \vec{a}_z \text{ V/m}$

Problem 3.57 $\vec{a}_n = 0.6 \vec{a}_x + 0.8 \vec{a}_y$

Let $\vec{E}_1 = E_{x1} \vec{a}_x + E_{y1} \vec{a}_y + E_{z1} \vec{a}_z$, $E_{y1} = 50 \text{ V/m}$

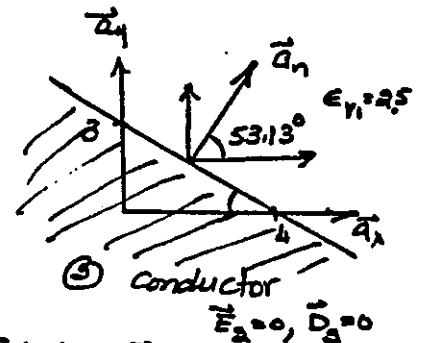
$\vec{a}_n \times (\vec{E}_1 - \vec{E}_2) = 0 \Rightarrow \vec{a}_n \times \vec{E}_1 = 0$

or $0.6 E_{y1} = 0.8 E_{x1} \Rightarrow E_{x1} = \frac{0.6}{0.8} \times 50 = 37.5 \text{ V/m.}$

and $E_{z1} = 0$

$\vec{a}_n \cdot \vec{D}_1 = \rho_s \Rightarrow \rho_s = (0.6 \vec{a}_x + 0.8 \vec{a}_y) \cdot (37.5 \vec{a}_x + 50 \vec{a}_y) 2.5 \epsilon_0$
 $= 1.38 \text{ nC/m}^2$

$\vec{E}_1 = 37.5 \vec{a}_x + 50 \vec{a}_y \text{ V/m.}$



Problem 3.58

$C = C_1 + C_2 + C_3 = \frac{A}{d} \epsilon_0 [\epsilon_{r1} + \epsilon_{r2} + \epsilon_{r3}]$
 $= \frac{100 \times 10^{-4}}{10^{-3}} \cdot \frac{10^{-9}}{36\pi} [2 + 3.6 + 9] = 1.29 \text{ nF}$

Problem 3.59 $C_1 = \frac{\epsilon_1 A_1}{d_1} = \frac{9 \times 10^{-9}}{36\pi} \cdot \frac{100 \times 10^{-4}}{0.5 \times 10^{-3}} = 1.592 \text{ nF}$

$C_2 = \frac{\epsilon_2 A_2}{d_2} = \frac{3.6 \times 10^{-9}}{36\pi} \cdot \frac{100 \times 10^{-4}}{0.5 \times 10^{-3}} = 0.637 \text{ nF}$

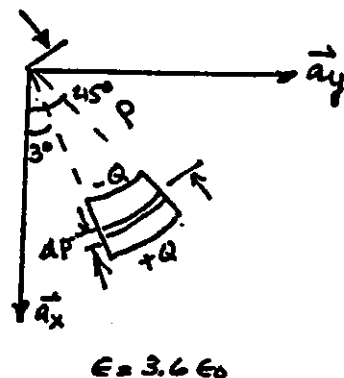
$C = \frac{C_1 C_2}{C_1 + C_2} = 0.455 \text{ nF}$

Problem 3.60 On a per-unit length basis:

$$C_1 = \frac{\epsilon_1 \pi}{\ln(b/a)} = \frac{\pi \times 5 \times 10^{-9}}{36\pi \ln(1.5)} = 0.343 \text{ nF/m}$$

$$C_2 = \frac{\pi \epsilon_2}{\ln(b/a)} = \frac{\pi \times 25 \times 10^{-9}}{36\pi \ln(1.5)} = 0.171 \text{ nF/m}, \quad C = C_1 + C_2 = 0.514 \text{ nF/m}$$

When $L=10$, $C_T = 5.14 \text{ nF}$



Problem 3.61 At a distance P , the area is

$$A = \int_0^{2\pi} P d\phi \int_0^{0.3} dz = 26.18 \times 10^{-3} P \text{ m}^2$$

$$\frac{1}{C} = \int \frac{d\ell}{\epsilon A} = \int_0.1^{0.3} \frac{dP}{26.18 \times 10^{-3} P \epsilon}$$

$$= \frac{38.2}{\epsilon} \ln(3) = 1.318 \times 10^{12} \Rightarrow C = 0.76 \text{ PF}$$

Problem 3.62 $C_1 = \frac{4\pi\epsilon_1 ac}{c-a} = \frac{4\pi \times 5 \times 10^{-9} \times 10 \times 20 \times 10^{-4}}{36\pi \times 10 \times 10^{-2}} = 111.11 \text{ PF}$

$$C_2 = \frac{4\pi\epsilon_2 bc}{b-c} = \frac{4\pi \times 10 \times 10^{-9} \times 20 \times 30 \times 10^{-4}}{36\pi \times 10 \times 10^{-2}} = 666.67 \text{ PF}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = 95.24 \text{ PF}$$

Problem 3.63 Outside: $\nabla^2 V = 0 \Rightarrow \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial V_o}{\partial r}) = 0 \Rightarrow V_o = -\frac{C_1}{r} + C_2$

Note: $P = P_0$ As $r \rightarrow \infty$, $V_o \rightarrow 0 \Rightarrow C_2 = 0$. Thus, $V_o(r) = -\frac{C_1}{r}$, $\vec{E}_o = -\nabla V_o = -\frac{C_1}{r^2} \vec{a}_r$

Inside, $r < b$: $\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial V_i}{\partial r}) = -\frac{P}{\epsilon_0} \Rightarrow \frac{d}{dr} (r^2 \frac{\partial V_i}{\partial r}) = -\frac{P}{\epsilon_0} r^2$

or $r^2 \frac{dV_i}{dr} = -\frac{P}{3\epsilon_0} r^3 + C_3 \Rightarrow \frac{dV_i}{dr} = -\frac{P}{3\epsilon_0} r + \frac{C_3}{r^2}$

Since $\vec{E}_i = -\nabla V_i = -\vec{a}_r (\frac{\partial V_i}{\partial r}) = -(\frac{P}{3\epsilon_0} r + \frac{C_3}{r^2}) \vec{a}_r$. As $r \rightarrow 0$, \vec{E}_i can't be ∞ . Thus, $C_3 = 0$

Thus, $\vec{E}_i = \frac{Pr}{3\epsilon_0} \vec{a}_r$. At $r=b$, $D_{n1} = D_{n2} \Rightarrow \epsilon_0 \frac{Pb}{3\epsilon_0} = -\epsilon_0 \frac{C_1}{b^2} \Rightarrow C_1 = -\frac{Pb^3}{3\epsilon_0}$

Hence: $\frac{dV_i}{dr} = -\frac{Pr}{3\epsilon_0} \Rightarrow V_i = -\frac{Pr^2}{6\epsilon_0} + C_4$. At $r=b$, $V_i = V_o \Rightarrow -\frac{Pb^2}{6\epsilon_0} + C_4 = -\frac{Pb^2}{3\epsilon_0}$

Hence $C_4 = \frac{Pb^2}{2\epsilon_0}$. Thus, $V_i = -\frac{Pr^2}{6\epsilon_0} + \frac{Pb^2}{2\epsilon_0} = \frac{P}{2\epsilon_0} [b^2 - \frac{r^2}{3}]$, $\vec{E}_i = \frac{Pr}{3\epsilon_0} \vec{a}_r$

and $V_o = \frac{Pb^3}{3\epsilon_0 r}$ and $\vec{E}_o = \frac{Pb^3}{3\epsilon_0} \cdot \frac{1}{r^2} \vec{a}_r$

Problem 3.64 $\nabla^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial z^2} = 0$ or $V = a_1 z + a_2$

At $z=0$ $V(0) = -100 \Rightarrow a_2 = -100$

At $z=0.04$ m, $V(0.04) = 100 \Rightarrow a_1 = 5000$

Hence $V(z) = 5000z - 100$

$\vec{E} = -\nabla V = -5000 \vec{a}_z$ V/m $\vec{D} = \epsilon_0 \vec{E} = -44.21 \vec{a}_z$ nC/m²

$\rho_{s+} = 44.21$ nC/m² and $\rho_{s-} = -44.21$ nC/m²

$z=4$ cm $\frac{100V}{-100V}$ ρ_{s+}

$z=0$ $\frac{-100V}{-100V}$ ρ_{s-}

Problem 3.65 $\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V}{\partial \rho} \right) = 0 \Rightarrow$

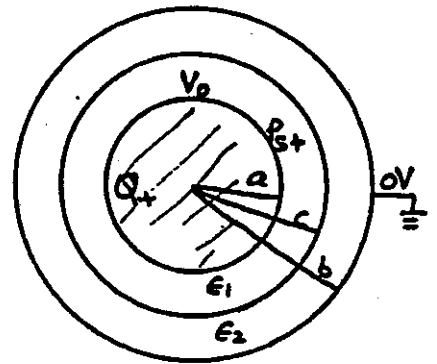
Region-1: $V_1 = A \ln \rho + B$ $a \leq \rho \leq c$

Region-2: $V_2 = G \ln \rho + H$ $c \leq \rho \leq b$

$V_1 = V_0$ at $\rho = a$ and $V_2 = 0$ at $\rho = b \Rightarrow$

$V_1 = A \ln(\rho/a) + V_0$ and $V_2 = G \ln(\rho/b)$

$E_{\rho 1} = -\frac{A}{\rho}$, $E_{\rho 2} = -\frac{G}{\rho}$



$V_1 = V_2$ at $\rho = c$
and $D_{\rho 1} = D_{\rho 2}$ at $\rho = c$

$D_{\rho 1} = -\frac{\epsilon_1 A}{\rho}$, $D_{\rho 2} = -\frac{\epsilon_2 G}{\rho}$, $D_{\rho 1} = D_{\rho 2} |_{\rho=c} \Rightarrow A = \frac{\epsilon_2}{\epsilon_1} G$ ①

$V_1 = V_2$ at $\rho = c \Rightarrow V_0 + A \ln(c/a) = G \ln(c/b)$ ②

From ① and ②: $G = \frac{\epsilon_1 V_0}{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)}$, $A = \frac{\epsilon_2 V_0}{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)}$

Hence, $V_1 = V_0 \left[\frac{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(\rho/c)}{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)} \right]$ and $V_2 = \frac{V_0 \epsilon_1 \ln(\rho/b)}{\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)}$

$\vec{E}_1 = \frac{-\epsilon_2 V_0 \vec{a}_\rho}{\rho [\epsilon_1 \ln(c/b) + \epsilon_2 \ln(a/c)]} = \frac{\epsilon_2 V_0 \vec{a}_\rho}{\rho [\epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)]}$

$\vec{E}_2 = \frac{\epsilon_1 V_0 \vec{a}_\rho}{\rho [\epsilon_1 \ln(b/c) + \epsilon_2 \ln(c/a)]}$, $\rho_{s+} = \vec{D}_1 \cdot \vec{a}_\rho |_{\rho=a} = \frac{\epsilon_1 \epsilon_2 V_0}{a [\epsilon_1 \ln(b/c) + \epsilon_2 \ln(a/c)]}$

$Q_+ = 2\pi a \rho_{s+} \Rightarrow C = \frac{Q_+}{V_0} = \frac{2\pi \epsilon_1 \epsilon_2}{\epsilon_1 \ln(b/c) + \epsilon_2 \ln(a/c)}$ F/m.
(per unit length)

Problem 3.66 Substitute $a=0.1\text{ m}$, $b=0.2\text{ m}$, $c=0.15\text{ m}$, $\epsilon_r=3$, and $\epsilon_{r2}=9$ in the solution of Problem 3.65 and obtain

$$V_1 = -199.458 \ln(P/a) + 100, \quad V_2 = -66.486 \ln(P/b) \quad V_0 = 100\text{ V}$$

$$\vec{E}_1 = \frac{199.458}{P} \vec{a}_P, \quad \vec{D}_1 = \frac{5.291}{P} \vec{a}_P \text{ nC/m}^2 \quad P_{S+}|_{P=a} = 52.91 \text{ nC/m}^2$$

$$\vec{E}_2 = \frac{66.486}{P} \vec{a}_P, \quad \vec{D}_2 = \frac{5.291}{P} \vec{a}_P \text{ nC/m}^2 \quad P_{S-}|_{P=b} = 26.46 \text{ nC/m}^2$$

$$Q_+ = 2\pi a P_{S+}(1) = 32.244 \text{ nC/m}$$

$$C = \frac{Q_+}{V_0} = 322.44 \text{ pF/m}$$

Problem 3.67 $\nabla^2 V = 0 \quad \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial V}{\partial r}) = 0 \Rightarrow V = -\frac{C_1}{r} + C_2$

at $r=b$, $V=0 \Rightarrow C_2 = \frac{C_1}{b}$ Thus, $V = C_1 [\frac{1}{b} - \frac{1}{r}]$

at $r=a$, $V=V_0 \Rightarrow C_1 = \frac{V_0}{\frac{1}{b} - \frac{1}{a}} \Rightarrow V = \frac{-V_0}{\frac{1}{b} - \frac{1}{a}} (\frac{1}{b} - \frac{1}{r}) = \frac{V_0 ab}{(b-a)r} - \frac{V_0 a}{b-a}$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r} \vec{a}_r = \frac{V_0}{b-a} \cdot \frac{1}{r^2} \vec{a}_r = \frac{V_0 ab}{(b-a)r^2} \vec{a}_r, \quad \vec{D} = \frac{V_0 \epsilon ab}{(b-a)r^2} \vec{a}_r$$

$$P_{S+}|_{r=a} = \frac{V_0 \epsilon b}{a(b-a)}, \quad Q_+ = 4\pi a^2 P_{S+} = \frac{4\pi a b \epsilon V_0}{b-a}$$

$$C = \frac{Q_+}{V_0} = \frac{4\pi \epsilon ab}{b-a} = \frac{4\pi \epsilon}{\frac{1}{a} - \frac{1}{b}}$$

Problem 3.68 Substitute $a=0.05\text{ m}$, $b=0.1\text{ m}$, $V_0=500\text{ V}$ and $\epsilon=9\epsilon_0$ in Problem 3.67 and obtain

$$V(r) = \frac{50}{r} - 500, \quad \vec{E} = \frac{50}{r^2} \vec{a}_r, \quad \vec{D} = \frac{3.98}{r^2} \vec{a}_r \text{ nC/m}^2$$

$$P_{S+} = \frac{3.98 \times 10^{-9}}{(0.05)^2} = 1.59 \text{ } \mu\text{C/m}^2, \quad Q_+ = 1.59 \times 10^{-6} \times 4\pi (0.05)^2 = 50 \text{ nC}$$

$$C = \frac{Q_+}{V_0} = 100 \text{ pF}$$

Problem 3.69 $\nabla^2 V = -\frac{A}{\epsilon P} \Rightarrow \frac{1}{P} \frac{\partial}{\partial P} \left(P \frac{\partial V}{\partial P} \right) = -\frac{A}{\epsilon P}$

Integrating $\frac{\partial V}{\partial P} = -\frac{A}{\epsilon} + \frac{C_1}{P}$ and $V = -\frac{A}{\epsilon} P + C_1 \ln P + C_2$ (1)

At $P=b, V=0 \Rightarrow C_2 = \frac{Ab}{\epsilon} - C_1 \ln b$ or $V = -\frac{A}{\epsilon} P + C_1 \ln(P/b) + \frac{Ab}{\epsilon}$

At $P=a, V=V_0 \Rightarrow C_1 = \frac{\frac{A}{\epsilon}(b-a) - V_0}{\ln(b/a)}$ and $C_2 = \frac{Ab}{\epsilon} - \left[\frac{A(b-a) - V_0}{\epsilon \ln(b/a)} \right] \frac{\ln b}{\ln(b/a)}$

Thus, $V = -\frac{AP}{\epsilon} + \frac{A}{\epsilon} b + \left[\frac{A(b-a) - V_0}{\epsilon \ln(b/a)} \right] \frac{\ln(P/b)}{\ln(b/a)}$

$\vec{E} = -\nabla V = -\frac{\partial V}{\partial P} \vec{a}_P = \left[\frac{A}{\epsilon} - \frac{A(b-a) - V_0}{P \ln(b/a)} \right] \vec{a}_P$, $\vec{D} = \epsilon \vec{E} = \left[A - \frac{A(b-a) - \epsilon V_0}{P \ln(b/a)} \right] \vec{a}_P$

$P_{st}|_{P=a} = A - \frac{A(b-a) - \epsilon V_0}{a \ln(b/a)}$, $Q_+ = 2\pi a P_{st} = 2\pi a \left[A - \frac{A(b-a) - \epsilon V_0}{a \ln(b/a)} \right]$

Thus, $C = \frac{Q_+}{V_0} = \frac{2\pi a}{V_0} \left[A - \frac{A(b-a) - V_0 \epsilon}{a \ln(b/a)} \right]$, when $A \rightarrow 0$ $C = \frac{2\pi \epsilon}{\ln(b/a)}$

Problem 3.70

$\nabla^2 V = 0 \Rightarrow \frac{1}{P^2} \frac{\partial^2 V}{\partial \phi^2} = 0 \Rightarrow$

$V = C_1 \phi + C_2$ at $\phi = 0, V = 0 \Rightarrow C_2 = 0$

At $\phi = \frac{\pi}{6}, V = 10,000 \Rightarrow C_1 = 19098.59$

Hence, $V = 19098.59 \phi$

$\vec{E} = -\nabla V = -\frac{1}{P} \frac{\partial V}{\partial \phi} \vec{a}_\phi = -\frac{19098.59}{P} \vec{a}_\phi$, $\vec{D} = \epsilon_0 \vec{E} = -\frac{168.87}{P} \vec{a}_\phi$ nc/m²

$P_{st} = \frac{168.87}{P}$ nc/m². The charge on the top-plate per-unit length in the z-direction is

$Q_+|_{\text{per-unit length}} = \int_{0.1}^{0.2} \frac{168.87 \times 10^9}{P} dP (1) = 117.05$ nc/m

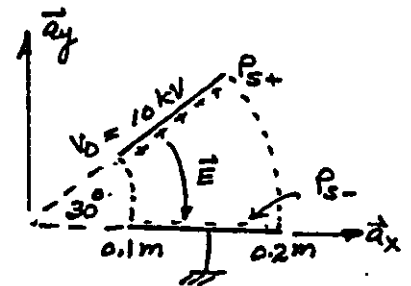
Thus, $C = \frac{Q_+|_{\text{per-unit length}}}{V_0} = 11.71$ pF/m

$W = \frac{1}{2} \vec{D} \cdot \vec{E} = \frac{1.613}{P^2}$ mJ/m³

$W = \int_{0.1}^{0.2} \frac{1.613}{P^2} P dP \int_0^{\pi/6} d\phi (1) = 585.26$ μJ/m

or $W = \frac{1}{2} C V_0^2$

$= \frac{1}{2} \times 11.71 \times 10^{-12} \times 10,000^2$
 $= 585.26$ μJ/m



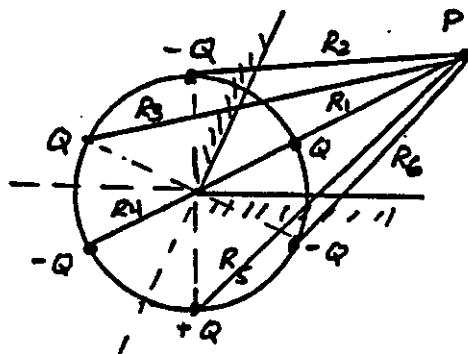
Problem 3.71 $P(7, 30^\circ, 10)$, Q is at $(5, 30^\circ, 10)$

$$R_1 = 2m, R_2 = 6.24m, R_3 = 10.44m$$

$$R_4 = 12m, R_5 = 10.44m, R_6 = 6.24m$$

$$V_P = \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{R_1} - \frac{1}{R_2} + \frac{1}{R_3} - \frac{1}{R_4} + \frac{1}{R_5} - \frac{1}{R_6} \right]$$

When $Q = 500 \text{ nC}$, $V_P \approx 1295 \text{ V}$



Problem 3.72

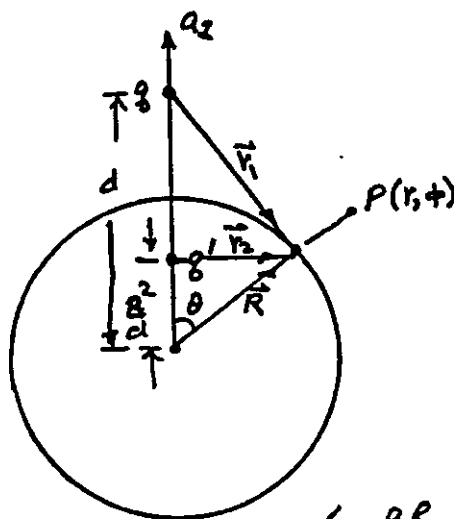
$$\vec{r}_1 = R\vec{a}_r - d\vec{a}_2$$

$$r_1^2 = \vec{r}_1 \cdot \vec{r}_1 = R^2 + d^2 - 2dR\vec{a}_r \cdot \vec{a}_2$$

$$= R^2 + d^2 - 2dR \cos\theta$$

$$r_1 = \sqrt{R^2 + d^2 - 2dR \cos\theta}$$

$$r_2 = \sqrt{R^2 + \frac{R^2}{d^2} - \frac{2R^2}{d} \cos\theta}$$



Potential at any point $P(r, \theta)$ is

$$V = \frac{\lambda}{4\pi\epsilon_0} \left[\frac{1}{(r^2 + d^2 - 2rd \cos\theta)^{1/2}} - \frac{\frac{R}{d}}{(r^2 + \frac{R^2}{d^2} - \frac{2rR}{d} \cos\theta)^{1/2}} \right] \quad \lambda' = \frac{\lambda R}{d}$$

$$\vec{E}|_{r=R} = -\nabla V|_{r=R} = \frac{\lambda}{4\pi\epsilon_0 R} \left[\frac{R^2 - d^2}{r^3} \right] \vec{a}_r \quad [\text{we need to compute } \vec{a}_r \text{ only}]$$

$$\rho_s|_{r=R} = \epsilon_0 \vec{E}|_{r=R} \cdot \vec{a}_r = \frac{\lambda}{4\pi R} \cdot \frac{R^2 - d^2}{r^3}$$

$$dQ = \rho_s ds = \frac{\lambda}{4\pi R} \frac{R^2 - d^2}{r^3} R^2 \sin\theta d\theta d\phi$$

$$Q = \frac{\lambda(R^2 - d^2)R^2}{4\pi R} \int_0^\pi \frac{1}{r^3} \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\lambda}{2d} [R^2 - d^2] \int_{d-R}^{d+R} \frac{dr_1}{r_1^3} = -\frac{\lambda}{2} \frac{R}{d}$$

$$r_1^2 = R^2 + d^2 - 2Rd \cos\theta$$

$$2r_1 dr_1 = +2Rd \sin\theta d\theta$$

$$dr_1 = \frac{Rd}{r_1} \sin\theta d\theta$$

$$\text{or } \sin\theta d\theta = \frac{r_1 dr_1}{Rd}$$

when $\theta = 0$ $r_1 = d - R$

when $\theta = \pi$ $r_1 = d + R$

